EE538 — Final Exam — Fall 2007
Mon, Dec 10, 8-10 am — RHPH 127 — Dec. 10, 2007

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators allowed!!
This test contains **five** problems.
Each of the **five** problems are equally weighted.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

Problem 1. This problem deals with the properties of deterministic autocorrelation.

(a) Consider the symmetric sequence below which is one for $-2 \leq m \leq 2$ and zero for $|m| > 2$. Is this is a valid autocorrelation sequence? Justify your answer.

$$r_{xx}[m] = u[m+2] - u[m-3]$$

(b) Consider the symmetric sequence below. Is this is a valid autocorrelation sequence? You need to explain your answer.

$$r_{xx}[m] = (3 - |m|)(u[m+2] - u[m-3])$$

(c) Let $r_{xx}[m]$ denote the autocorrelation sequence for the DT signal $x[n]$. Let $y[n] = x[n - n_o]$, where n_o is an integer. Let $r_{yy}[m]$ denote the autocorrelation sequence for the DT signal $y[n]$. Derive an expression relating $r_{yy}[m]$ and $r_{xx}[m]$. That is, how is $r_{yy}[m]$ related to $r_{xx}[m]$?

(d) Let $r_{xx}[m]$ denote the autocorrelation sequence for the DT signal $x[n]$. Let $y[n] = e^{j(\omega_0 n+\theta)}x[n]$, where ω_o is some frequency and θ is some phase value. Let $r_{yy}[m]$ denote the autocorrelation sequence for the DT signal $y[n]$. Derive an expression relating $r_{yy}[m]$ and $r_{xx}[m]$. That is, how is $r_{yy}[m]$ related to $r_{xx}[m]$?

(e) Consider that the signal $x[n]$ below, where $a = \frac{1}{2}$, is the input to the LTI system described by the difference equation in equation (2) below.

$$x[n] = a^n u[n] - \frac{1}{a} a^{n-1} u[n-1] \tag{1}$$

$$y[n] = \frac{1}{4} y[n-1] + x[n] \tag{2}$$

(i) Determine a closed-form analytical expression for the auto-correlation $r_{xx}[m]$ for $x[n]$, when $a = \frac{1}{2}$. (Hint: examine the Z-Transform of $x[n]$.)

(ii) Determine a closed-form analytical expression for the cross-correlation $r_{yx}[m]$ between the input and output.

(iii) Determine a closed-form analytical expression for the auto-correlation $r_{yy}[m]$ for the output $y[n]$.

Problem 2. [20 points]

Consider the discrete-time complex-valued random process defined for all n:

$$x[n] = A_1 e^{\jmath(\omega_1 n + \Theta_1)} + A_2 e^{\jmath(\omega_2 n + \Theta_2)}$$

where the respective frequencies, ω_1 and ω_2, of the two complex sinewaves are deterministic but unknown constants. The amplitudes, A_1 and A_2, and the constant D are also deterministic but unknown constants. Θ_1 and Θ_2 are independent random variables with each uniformly distributed over a 2π interval. The values of the autocorrelation sequence for $x[n]$, $r_{xx}[m] = E\{x[n]x^*[n-m]\}$, for three different lag values are given below.

$$r_{xx}[0] = 2, \quad r_{xx}[1] = 1 + j\left(1 + \frac{1}{\sqrt{2}}\right), \quad r_{xx}[2] = -1 + j,$$

(a) Determine the numerical values of ω_1 and ω_2. **You have to use what you've learned during the parametric spectral analysis portion of this course. You will be given no credit if you simply set up a system of equations to solve based on the form of $r_{xx}[m] = \sum_{i=1}^{p} A_i^2 e^{j\omega_i m}$ and solve this nonlinear system of equations.**

(b) Consider a first-order predictor

$$\hat{x}[n] = -a_1(1)x[n-1]$$

Determine the numerical values of the optimum predictor coefficient $a_1(1)$, and the numerical value of the corresponding minimum mean-square error $\mathcal{E}^1_{min}$.

(c) Consider a second-order predictor

$$\hat{x}[n] = -a_2(1)x[n-1] - a_2(2)x[n-2]$$

Determine the numerical values of the optimum predictor coefficients $a_2(1)$ and $a_2(2)$, and the numerical value of the corresponding minimum mean-square error $\mathcal{E}^2_{min}$.

(d) Consider a third-order predictor

$$\hat{x}[n] = -a_3(1)x[n-1] - a_3(2)x[n-2] - a_3(3)x[n-3]$$

Determine the numerical values of the optimum predictor coefficients $a_3(1)$, $a_3(2)$ and $a_3(3)$, and the numerical value of the corresponding minimum mean-square error $\mathcal{E}^3_{min}$.

Problem 3. [20 pts]

Consider an ARMA(1,1) process generated by passing the white noise random process $\nu[n]$, with $r_{\nu\nu}[m] = \delta[m]$, through the LTI system described by the difference equation below having one pole and one zero.

$$x[n] = -a_1 x[n-1] + b_0 \nu[n] + b_1 \nu[n-1]$$

You are given the first four values of the autocorrelation sqeuence for the output ARMA process $x[n]$.

$$\begin{aligned} r_{xx}[0] &= 24 \\ r_{xx}[1] &= 20 \\ r_{xx}[2] &= 40/3 \\ r_{xx}[3] &= ?? \end{aligned}$$

(a) Determine the numerical value of the ARMA model parameter a_1 in the difference equation above.

(b) Determine the numerical values of the optimum second-order linear prediction coefficients $a_2(1)$ and $a_2(2)$, and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(2)}$.

(c) Determine the numerical value of $r_{xx}[3]$.

(d) Determine the autocorrelation function $r_{bb}[m]$ for the sequence $b[n] = \{b_0, b_1\}$. Even though you were not given the values of b_0 and b_1, you are given enough information to nonetheless determine the numerical values of $r_{bb}[-1]$, $r_{bb}[0]$, and $r_{bb}[1]$. Find these three values.

(e) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$$

Problem 4. [20 points]
We wish to filter a data stream with a filter having the following impulse response

$$h[n] = e^{-j\frac{2\pi}{N}\ell n}\{u[n] - u[n-N]\}$$

where ℓ is an integer between 0 and $N-1$. Convolution with this impulse response may be effected via the following difference equation

$$y[n] = \sum_{k=0}^{N-1} e^{j\frac{-2\pi}{N}\ell k} x[n-k]$$

This implementation requires N multiplications and $N-1$ additions per output point.

(a) It can be shown that we can achieve exactly the same input-output (I/O) relationship via the following difference equation which requires only 1 multiplication and two additions per output point.

$$y[n] = a_1\, y[n-1] + x[n] - x[n-D]$$

Determine the values of a_1 and D in terms of ℓ and N so that this IIR system has exactly the same I/O relationship as the FIR filter above.

(b) Consider the specific case of $\ell = 2$ and $N = 4$:

(i) State the numerical values of a_1 and D for this case so that the FIR filter and the IIR filter have the same I/O relationship.

(ii) Plot the pole-zero diagram for the system. Show the region of convergence.

(iii) Is this a lowpass, bandpass, or highpass filter? Briefly explain why.

(iv) Plot the impulse response of the system (Stem plot).

(c) Consider the difference equation below:

$$y[n] = -y[n-1] + x[n] - x[n-4]$$

Let the input $x[n]$ be a stationary white noise process with variance $\sigma_x^2 = 1$.

(i) Determine the autocorrelation sequence $r_{yy}[m] = E\{y[n]y[n-m]\}$. State the numerical values of $r_{yy}[m]$ for $m = 0, 1, 2, 3$ – four answers needed. Is $r_{yy}[m] = 0$ for $m > 3$? Why or why not?

(ii) Determine the spectral density of $y[n]$, $S_{yy}(\omega)$, the DTFT of $r_{yy}[m] = E\{y[n]y[n-m]\}$. Plot $S_{yy}(\omega)$ for $-\pi < \omega < \pi$.

Problem 5. [20 points]

Consider the ARMA(1,1) process generated via the difference equation

$$x[n] = -a_1 x[n-1] + b_0 \nu[n] + b_1 \nu[n-1]$$

where $\nu[n]$ is a stationary white noise process with variance $\sigma_w^2 = 1$. The autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$ is given by the following closed-form expression which holds for m from $-\infty$ to ∞:

$$r_{xx}[m] = 30\left(\frac{2}{3}\right)^{|m|} - 6\delta[m]$$

(a) Consider that the power spectrum of the ARMA(1,1) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}^{(1)}}{|1 + a_1^{(2)} e^{-j\omega}|^2}$$

Determine the respective numerical values of the optimum second-order linear prediction coefficient $a_1^{(1)}$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(1)}$.

(b) Determine the numerical value of the coefficient a_1 in the difference equation above defining the LTI system that the white noise was passed through to generate the ARMA(1,1) process $x[n]$.

(c) With the value of a_1 determined from part (b), determine the deterministic autocorrelation sequence $r_{aa}[m]$ for the length two sequence $a[n] = \{1, a_1\}$. Determine the numerical values of the three nonzero values: $r_{aa}[-1]$, $r_{aa}[0]$, and $r_{aa}[1]$.

(d) Convolve $r_{aa}[m]$ with $r_{xx}[m]$ to form $r_{bb}[m]$. List the numerical values of $r_{bb}[m] = r_{xx}[m] * r_{aa}[m]$ for all m.

(e) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$$