EE538 — Final Exam — Fall 2006

On-Campus: 8-10 am — WTHR 172 — Dec. 15, 2006

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators **not** allowed!!
This test contains **five** problems.
Each of the **five** problems are equally weighted.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

Problem 1. [20 points]

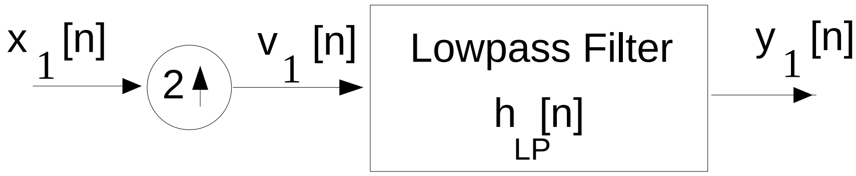

Figure 1.

Let $x_1[n]$ be the following DT signal.

$$x_1[n] = 2\left\{\frac{\sin(\frac{\pi}{6}n)}{\pi n}\right\}\cos\left(\frac{\pi}{2}n\right)$$

$x_1[n]$ is input to the system above, where the impulse response of the lowpass filter is

$$h_{LP}[n] = \left\{\frac{\sin(\frac{2\pi}{3}n)}{\pi n}\right\} + \left\{\frac{\sin(\frac{\pi}{3}n)}{\pi n}\right\}, \qquad -\infty < n < \infty,$$

The zero inserts may be mathematically described as $v_1[n] = \sum_{k=-\infty}^{\infty} x_1[k]\delta[n-2k]$.

(a) Plot the magnitude of the DTFT of the impulse response of the lowpass filter $h_{LP}[n]$, $H_{LP}(\omega)$, over $-\pi < \omega < \pi$. Show as much detail as possible.

(b) Plot the magnitude of the DTFT of the signal after the zero inserts $v_1[n]$, $V_1(\omega)$, over $-\pi < \omega < \pi$. Show as much detail as possible. The edge frequencies of any rectangularly shaped spectral components need to be clearly indicated.

(c) Plot the magnitude of the DTFT of the output $y_1[n]$, $Y_1(\omega)$, over $-\pi < \omega < \pi$. Show as much detail as possible. The edge frequencies of any rectangularly shaped spectral components need to be clearly indicated.

(d) The up-sampling by a factor of 2 in Figure 1 above can be efficiently done via the top half of the block diagram in Figure 2 at the top of the next page.

   (i) Provide an analytical expression for $h_{LP}^{(0)}[n] = h_{LP}[2n]$ for $-\infty < n < \infty$. Simplify. Plot the magnitude of the DTFT of $h_{LP}^{(0)}[n]$, $|H_{LP}^{(0)}(\omega)|$, over $-\pi < \omega < \pi$.

   (ii) Is $y_1^{(0)}[n] = x_1[n]$? Explain your answer.

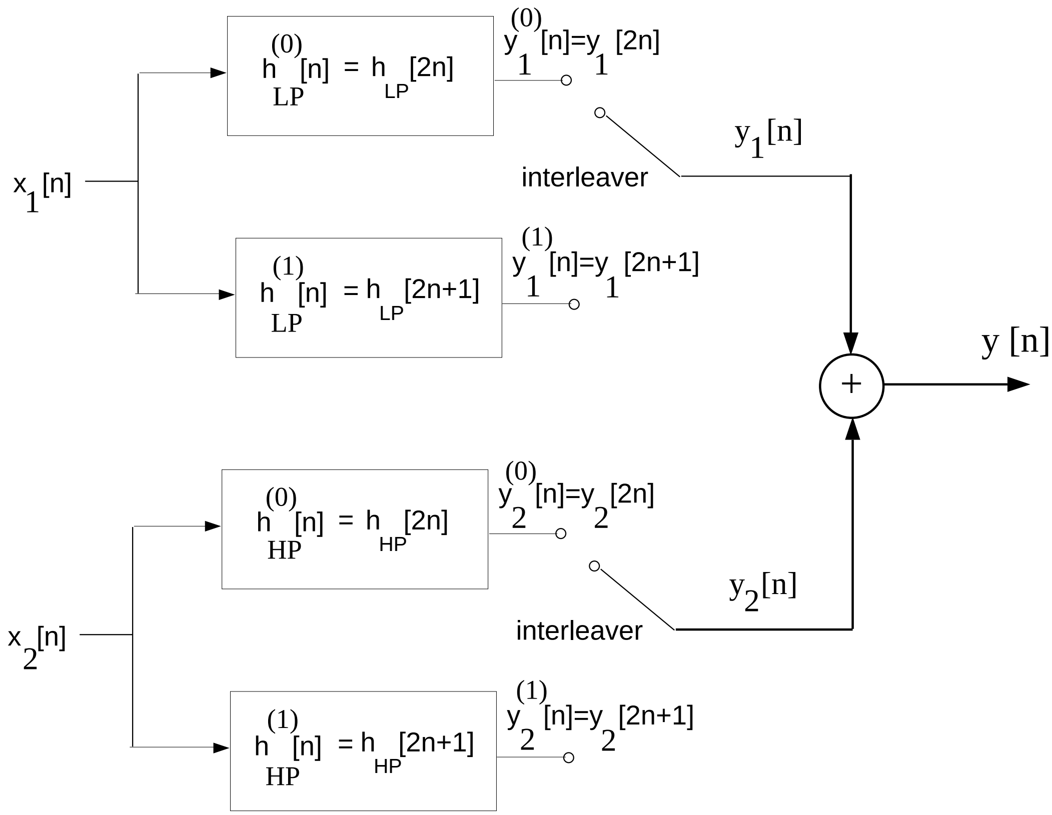

Figure 2.

Problem 2. Consider a second DT signal equal to a sum of sinwaves "turned on" for all time.

$$x_2[n] = 1 + (-j)^n + (j)^n$$

We desire to frequency division multiplex $x_1[n]$ (defined in Problem 2 on the previous page) and $x_2[n]$ directly above through digital subbanding as shown in Figure 2 above, where the highpass filter is

$$h_{HP}[n] = (-1)^n h_{LP}[n] = (-1)^n \left\{ \left\{ \frac{\sin(\frac{2\pi}{3}n)}{\pi n} \right\} + \left\{ \frac{\sin(\frac{\pi}{3}n)}{\pi n} \right\} \right\} \qquad -\infty < n < \infty,$$

The sum signal is $y[n] = y_1[n] + y_2[n]$, where $y_1[n]$ is the same $y_1[n]$ created in Problem 2.

(a) Plot the magnitude of the DTFT of the sum signal $y[n] = y_1[n] + y_2[n]$. Show as much detail as possible. The respective frequencies of any sinusoidal components need to be clearly indicated as well as the edge frequencies of any rectangularly shaped spectral components.

(b) Draw a block diagram of a system to recover $x_1[n]$ from the sum signal $y[n] = y_1[n] + y_2[n]$. The recovery of $x_1[n]$ must be done in a computationally efficient manner. You CANNOT use any decimators and you CANNOT use any modulators (no multiplication by a sinewave). Clearly specify all the quantities in your block diagram.

(c) Draw a block diagram of a system to recover $x_2[n]$, from the sum signal $y[n] = y_1[n] + y_2[n]$. The same rules apply as those stated in part (b) above.

Problem 3. [20 pts]

Consider an ARMA process generated by passing the white noise random process $\nu[n]$ through the LTI system described by the difference equation below having two poles and one zero.

$$x[n] = -a_1 x[n-1] - a_2 x[n-2] + b_0 \nu[n] + b_1 \nu[n-1]$$

You are given the first four values of the autocorrelation sqeuence for the output ARMA process $x[n]$.

$$r_{xx}[0] = 35$$
$$r_{xx}[1] = 13$$
$$r_{xx}[2] = 5$$
$$r_{xx}[3] = 2$$

(a) Determine the respective numerical values of the ARMA model parameters a_1 and a_2 in the difference equation above.

(b) Determine the numerical value of the optimum first-order linear prediction coefficient $a_1(1)$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(1)}$.

Problem 4. [20 pts]

Consider a discrete-time random process composed of TWO complex sinewaves in white noise expressed as

$$x[n] = A_1 \, e^{j(\omega_1 n + \Phi_1)} \; + \; A_2 \, e^{j(\omega_2 n + \Phi_2)} + \nu[n]$$

where the amplitudes A_1 and A_2 are unknown but deterministic constants, the frequencies ω_1 and ω_2 are unknown but deterministic constants, and the respective phases Φ_1 and Φ_2 are i.i.d. random variables with a uniform distribution over $[0, 2\pi)$. The autocorrelation for the zero mean white noise random process $\nu[n]$ is $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \delta[m]$. The white noise process, $\nu[n]$, is statistically independent of the sinewave signals.

The autocorrelation sequence is defined as

$$r_{xx}[m] = E\{x[n]x^*[n-m]\}$$

For a specific set of amplitudes and frequencies (unknown to you), you are given the values of the autocorrelation sequence for lag values $m = 0$, $m = 1$, and $m = 2$. (Note: $r_{xx}[m]$ is generally nonzero for $m > 2$.)

$$\begin{aligned} r_{xx}[0] &= 4 \\ r_{xx}[1] &= -2 - j \\ r_{xx}[2] &= 1 \\ r_{xx}[3] &= ?? \end{aligned}$$

(a) Determine the respective numerical values of the two frequencies ω_1 and ω_2.

(b) Determine the numerical value of $r_{xx}[3]$.

(c) The spectral density, $S_{xx}(\omega)$, is the Discrete-Time Fourier Transform (DTFT) of $r_{xx}[m]$. Plot the magnitude of $S_{xx}(\omega)$ over $-\pi < \omega < \pi$.

(d) The random process $x[n]$ above is the input signal to the system described by the difference equation below. Plot the magnitude of $S_{yy}(\omega)$ for the output signal over $-\pi < \omega < \pi$.

$$y[n] = y[n-1] + x[n] - x[n-4]$$

Problem 5. [20 points]

Consider the ARMA(1,1) process generated via the difference equation

$$x[n] = a_1 x[n-1] + b_0 w[n] + b_1 w[n-1]$$

where $w[n]$ is a stationary white noise process with variance $\sigma_w^2 = 1$. The autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$ is given by the following closed-form expression which holds for m from $-\infty$ to ∞:

$$r_{xx}[m] = 6\left(\frac{1}{2}\right)^{|m|} - 2\delta[m]$$

(a) Consider that the power spectrum of the ARMA(1,1) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}^{(2)}}{|1 + a_1^{(2)} e^{-j\omega} + a_2^{(2)} e^{-j2\omega}|^2}$$

Determine the respective numerical values of the optimum second-order linear prediction coefficients $a_1^{(2)}$ and $a_2^{(2)}$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(2)}$.

(b) Determine the numerical value of the coefficient a_1 in the difference equation above defining the LTI system that the white noise was passed through to generate the ARMA(1,1) process $x[n]$.

(c) Determine the respective numerical values of the coefficients b_0 and b_1 in the difference equation above defining the LTI system that the white noise was passed through to generate the ARMA(1,1) process.

(d) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$$