EE538 **Final Exam** **Fall 2005**
3:20 pm -5:20 pm **PHYS 223** **Dec. 17, 2005**

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators **ARE** allowed!!
This test contains **five** problems.
Each of the **five** problems are equally weighted.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

Problem 1. [20 points] Consider the transmission of a pulse amplitude-modulated signal

$$x(t) = \sum_{k=-\infty}^{\infty} b[k]p(t - kT_o)$$

where $b[n]$ is a zero-mean, stationary, discrete-time random process with $E\{b[n]b[n-m]\} = \delta[m]$. For any discrete-time n, $b[n]$ is a random variable equal to either "+1" or "-1" with equal probability. $1/T_o$ is the bit rate and $p(t)$ is the pulse symbol waveform below

$$p(t) = \frac{1}{T_o}\{T_o - |t|\}\ \{u(t + T_o) - u(t - T_o)\}$$

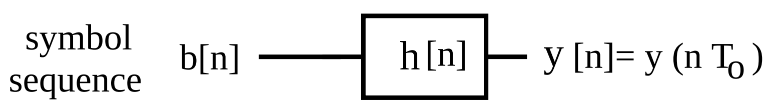
At the receiver, $x(t)$ arrives by both a direct path and a multipath reflection having the same strength as the direct path but at a delay of T_o and phase-shifted by θ. Denoting continuous time convolution as $*$, the received signal, $y(t)$, may be modeled as:

$$y(t) = x(t) * g(t)$$

where $g(t)$ is described below using $\delta(t)$ to denote the Dirac Delta function.

$$g(t) = \delta(t) + e^{j\frac{\pi}{2}}\delta\left(t - T_o\right) \tag{1}$$

Sampling $y(t)$ at the bit rate, $F_s = \frac{1}{T_o}$, it is easily shown that the resulting sequence $y[n] = y(nT_o)$ may be modeled as having been generated by the following discrete-time system.



Determine a closed-form expression for the autocorrelation sequence $r_{yy}[m]$ for $y[n] = y(nT_o)$ **AND** plot the corresponding spectral density $S_{yy}(\omega) = \sum_{n=-\infty}^{\infty} r_{yy}[m]e^{-j\omega m}$ over $-\pi \leq \omega \leq \pi$. $S_{yy}(\omega)$ will be exactly equal to zero for one specific value of ω. Determine this frequency.

GIVEN NOBLE's FIRST IDENTITY TO USE IN PROBLEM 2. If $E(\omega)$ in Figure 3(b) satsifies $E(\omega) = G(M\omega)$, the I/O relationship of the system in Figure 3(b) is exactly the same as the I/O relationship of the system in Figure 3(a). You are GIVEN this as Noble's First Identity to use in Problem 2 below.

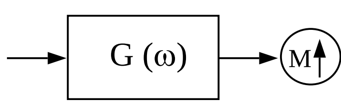


Figure 3(a).

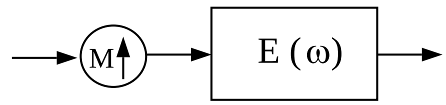


Figure 3(b).

Problem 2. [20 points]

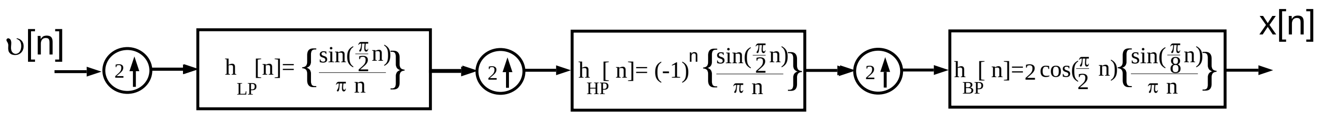


Figure 4(a).

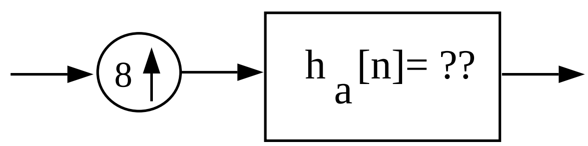


Figure 4(b).

(a) Determine the impulse response $h_a[n]$ in Figure 4(b) so that the I/O relationship of the system in Figure 4(b) is exactly the same as the I/O relationship of the system in Figure 4(a). Plot the magnitude of the DTFT of $h_a[n]$ over $-\pi < \omega < \pi$.

(b) Given that the input $\nu[n]$ is a zero-mean, stationary, discrete-time real-valued random process with $E\{\nu[n]\nu[n-m]\} = \delta[m]$. Determine the autocorrelation sequence $r_{xx}[m]$ for the output random process $x[n]$ and plot the corresponding spectral density $S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[m]e^{-j\omega m}$ over $-\pi \leq \omega \leq \pi$.

(c) $x[n]$ is predicted in terms of one past output value according to $\hat{x}[n] = -a_1(1)x[n-1]$. What is the value of $a_1(1)$ that minimizes the mean square value of the prediction error? What is the numerical value of the minimum mean-square prediction error?

(d) $x[n]$ is predicted in terms of two past values according to $\hat{x}[n] = -a_2(1)x[n-1] - a_2(2)x[n-2]$. What are the values of $a_2(1)$ and $a_2(1)$ that minimize the mean square value of the prediction error? What is the value of the minimum mean-square prediction error? *NOTE:* $\sqrt{2}$ and π can be carried along as constants in your answer.

Problem 3. [20 pts]

A discrete-time system of the form $y[n] = -a_1 y[n-1] - a_2 y[n-2] - b_0 x[n] - b_1 x[n-1]$ has a corresponding impulse response of the form

$$h[n] = c_1\ (p_1)^n\ u[n]\ +\ c_2\ (p_2)^n\ u[n]$$

where $u[n]$ is the discrete-time unit step sequence. You are given the first four values of this infinite length impulse response (IIR) below. (Note: $h[n]$ is generally nonzero for $n > 3$.)

$$\begin{aligned} h[0] &= 35 \\ h[1] &= 13 \\ h[2] &= 5 \\ h[3] &= 2 \end{aligned}$$

(a) Determine the respective numerical values of the two poles p_1 and p_2.

(b) Determine the numerical values of the two multiplicative constants c_1 and c_2.

(c) Draw a block diagram of a linear time-invariant (LTI) discrete-time system whose impulse response is $h[n]$. You are only allowed to have TWO unit delay blocks.

NOTE: Trying to set up four equations in four unknowns in the straightforward manner that first comes to mind leads to a system of equations involving a cubic equation in two unknowns. This is NOT the way to solve the problem and does not use any signal processing knowledge or background. You will not receive any points for taking this approach.

Problem 4. [20 pts]

Consider a discrete-time random process composed of TWO complex sinewaves expressed as

$$x[n] = A_1\, e^{j(\omega_1 n + \Phi_1)} \;+\; A_2\, e^{j(\omega_2 n + \Phi_2)}$$

where the amplitudes A_1 and A_2 are unknown but deterministic constants, the frequencies ω_1 and ω_2 are unknown but deterministic constants, and the respective phases Φ_1 and Φ_2 are i.i.d. random variables with a uniform distribution over $[0, 2\pi)$.

The autocorrelation sequence is defined as

$$r_{xx}[m] = E\{x[n]x^*[n-m]\}$$

For a specific set of amplitudes and frequencies (unknown to you), you are given the values of the autocorrelation sequence for lag values $m = 0$, $m = 1$, and $m = 2$. (Note: $r_{xx}[m]$ is generally nonzero for $m > 2$.)

$$\begin{aligned} r_{xx}[0] &= 2 \\ r_{xx}[1] &= \sqrt{2}j \\ r_{xx}[2] &= 0 \\ r_{xx}[3] &= ?? \end{aligned}$$

(a) Determine the respective numerical values of the two frequencies ω_1 and ω_2.

(b) Determine the numerical value of $r_{xx}[3]$.

(c) The spectral density, $S_{xx}(\omega)$, is the Discrete-Time Fourier Transform (DTFT) of $r_{xx}[m]$. Plot $S_{xx}(\omega)$ over $-\pi < \omega < \pi$.

NOTE: Again, trying to set up four equations in four unknowns in the straightforward manner that first comes to mind is NOT the way to solve the problem. You will not receive any points for taking this approach.

Problem 5. [20 points]

Consider the ARMA(2,2) process generated via the difference equation

$$x[n] = \frac{1}{4}x[n-2] + \nu[n] - 4\nu[n-2]$$

where $\nu[n]$ is a stationary white noise process with variance $\sigma_\nu^2 = 2$.

(a) Determine a simple, closed-form expression for the autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$ which holds for $-\infty < m < \infty$.

(b) Determine a simple, closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$. Plot $S_{xx}(\omega)$ for $-\pi < \omega < \pi$.

(c) Consider that the power spectrum of the ARMA(2,2) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}^{(1)}}{|1 + a_1(1)e^{-j\omega}|^2}$$

Determine the respective numerical values of the optimum first-order linear prediction coefficient $a_1(1)$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(1)}$.

(d) Consider that the power spectrum of the ARMA(2,2) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}(2)}{|1 + a_2(1)e^{-j\omega} + a_2(2)e^{-j2\omega}|^2}$$

Determine the respective numerical values of the optimum second-order linear prediction coefficients $a_1(2)$ and $a_2(2)$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(2)}$.