EE538	**Final Exam**	**Fall 2003**
8-10 am	**MSEE B012**	**Dec. 15, 2003**

Cover Sheet

Test Duration: 120 minutes.
Open Book but Closed Notes.
Calculators **not** allowed
This test contains **six** problems.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

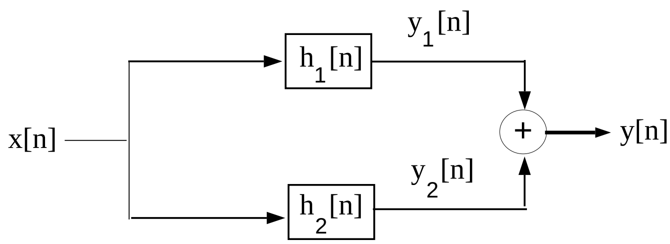
No.	**Topic(s) of Problem**	**Points**
1.	Convolution; LTI systems; Pole-Zero Diagrams	30
2.	Digital Upsampling	30
3.	Basics of AR spectral estimation; AR & linear prediction relationship	30
4.	Sampling of DTFT and Time-Domain Aliasing	30
5.	Autoregressive spectral estimation; basics of AR, MA, and ARMA processes.	40
6.	AR and ARMA spectral estimation; ZT & DTFT Relationship	40

Problem 1. [30 points]

Consider the causal, second-order difference equation below where it is noted that $.9025 = .95^2$. Also, you need to make the approximation $\frac{1}{.95} \approx 1.05$.

$$y[n] = -0.95\ y[n-1] - 0.9025\ y[n-2] + x[n] +\ x[n-1] +\ x[n-2]$$

Consider implementing this second-order difference equation as two first-order systems in parallel (one pole each) as shown in the diagram.



(a) Determine and write the first-order difference equation for each of the two first-order systems in parallel. The upper first-order system has impulse response $h_1[n]$ and is described by the difference equation

$$y_1[n] = a_1^{(1)}\ y_1[n-1] + b_0^{(1)} x[n] + b_1^{(1)}\ x[n-1]$$

The lower first-order system has impulse response $h_2[n]$ and is described by the difference equation

$$y_2[n] = a_1^{(2)}\ y_2[n-1] + b_0^{(2)} x[n] + b_1^{(2)}\ x[n-1]$$

Determine the numerical values of $a_1^{(i)}$, $b_0^{(i)}$, and $b_1^{(i)}$, $i = 1, 2$ – six values total.

(b) For EACH of the two first-order systems, $i = 1, 2$, do the following:

(i) Plot the pole-zero diagram.

(ii) State and plot the region of convergence for $H_i(z)$.

(iii) Determine the DTFT of $h_i[n]$ and plot the magnitude $|H_i(\omega)|$ over the interval $-\pi < \omega < \pi$ showing as much detail as possible. In particular, explicitly point out if there are any values of ω for which $|H_i(\omega)|$ is exactly zero.

Problem 2. [30 points]

The analog signal $x_a(t)$ is reconstructed from its samples $x[n] = x_a(nT_s)$ according to the following equation

$$x_r(t) = \sum_{k=-\infty}^{\infty} x[k]\, g(t - kT_s) \qquad \text{where: } g(t) = \begin{cases} 0.5 + 0.5\cos\left(\pi \frac{t}{T_s}\right) & \text{for } |t| < T_s \\ 0 & \text{for } |t| > T_s \end{cases}$$

Samples of the reconstructed signal at L times the original sampling rate may be obtained via the following discrete-time system.

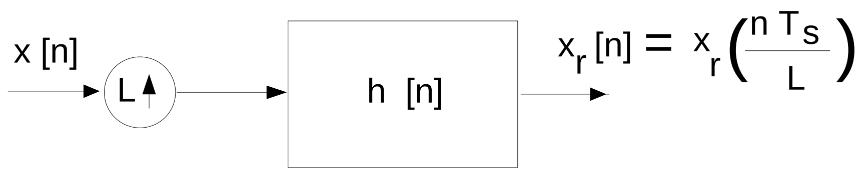


Figure 1.

Your primary task in this problem is to determine the appropriate filter impulse response $h[n]$ for different values of L so that the output of the system above is what you would have obtained if you had sampled the reconstructed signal at L times the original sampling rate as specified. **NOTE:** Correct answer for $h[n]$ is different for each part (for each value of L.)

(a) For the case of $L = 1$, write a closed-form expression for the filter $h[n]$. Plot the magnitude of the DTFT of $h[n]$, $H(\omega)$, over $-\pi < \omega < \pi$.

(b) For the case of $L = 2$, write a closed-form expression for the filter $h[n]$. Plot the magnitude of the DTFT of $h[n]$, $H(\omega)$, over $-\pi < \omega < \pi$. The case of $L = 2$ in Figure 1 may be efficient implementated as in the block diagram in Figure 2 below.

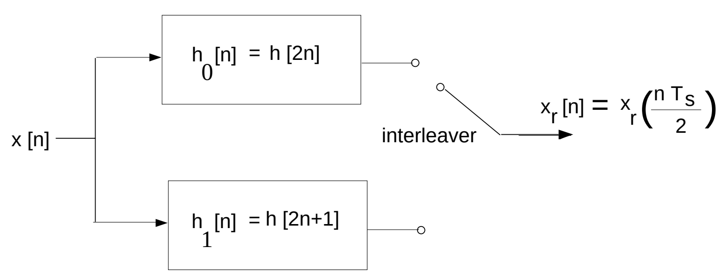


Figure 2.

Using your answer for $h[n]$ for this part (b), do the following:

(i) Write an expression for $h_0[n] = h[2n]$. Plot the magnitude of the DTFT of $h_0[n]$, $|H_0(\omega)|$, over $-\pi < \omega < \pi$.

(ii) Write an analytical expression for $h_1[n] = h[2n + 1]$. Plot the magnitude of the DTFT of $h_1[n]$, $|H_1(\omega)|$, over $-\pi < \omega < \pi$.

(c) For the case of $L = 4$, write a closed-form expression for the filter $h[n]$. Plot the magnitude of the DTFT of $h[n]$, $H(\omega)$, over $-\pi < \omega < \pi$.

Problem 3. [30 points]

Consider the autoregressive AR(4) process generated via the difference equation

$$x[n] = \frac{1}{2}x[n-1] - \frac{1}{4}x[n-2] + \frac{1}{8}x[n-3] - \frac{1}{16}x[n-4] + \nu[n]$$

where $\nu[n]$ is a stationary, white noise process with variance $\sigma_w^2 = 3$.

(a) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m] = E\{x[n]x[n-m]\}$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega}$$

(b Consider the fourth-order predictor

$$\hat{x}[n] = -a_4(1)x[n-1] - a_4(2)x[n-2] - a_4(3)x[n-3] - a_4(4)x[n-4]$$

Determine the numerical values of the optimum predictor coefficients $a_4(1)$, $a_4(2)$, $a_4(3)$, and $a_4(4)$**and** compute the corresponding minimum mean-square error.

(b Consider the fifth-order predictor

$$\hat{x}[n] = -a_5(1)x[n-1] - a_5(2)x[n-2] - a_5(3)x[n-3] - a_5(4)x[n-4] - a_5(5)x[n-5]$$

Determine the numerical values of the optimum predictor coefficients $a_5(1)$, $a_5(2)$, $a_5(3)$, $a_5(4)$ and $a_5(5)$ **and** compute the corresponding minimum mean-square error.

Problem 4. [30 points]

(a) Let $X_8(k) = X(2\pi k/8)$, where $X(\omega)$ is the DTFT of the sequence

$$x[n] = (0.9)^n e^{j\frac{\pi}{7}n}u[n] \overset{DTFT}{\longleftrightarrow} X(\omega) = \frac{1}{1 - 0.9e^{j\frac{\pi}{7}}e^{-j\omega}}$$

That is, $X_8(k)$ is the frequency domain sequence obtained by sampling $X(\omega)$ at $N = 8$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 8-pt inverse DFT of $X_8(k)$ may be expressed as

$$x_8[n] = \sum_{\ell=-\infty}^{\infty} x[n-\ell 8]\{u[n]-u[n-8]\} \underset{8}{\overset{DFT}{\longleftrightarrow}} X_8(k) = \frac{1}{1 - 0.9e^{j\frac{\pi}{7}}e^{-j2\pi k/8}}; k = 0, 1, ..., 7$$

Determine a simple, closed-form expression for $x_8[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers. *Hint:*

$$\frac{1}{1 - (.9e^{j\frac{\pi}{7}})^8} = -0.2943 + j0.0832$$

(b) Consider normalizing $x_8[n]$ so that it's first value is one, $\tilde{x}_8[n] = x_8[n]/x_8[0]$, $n = 0, 1, ..., 7$. Compare $\tilde{x}_8[n]$ and $x[n]$ over $n = 0, 1, ..., 7$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

(c) Let $Y_8(k) = Y(2\pi k/8)$, where $Y(\omega)$ is the DTFT of the sequence

$$y[n] = 2\ (0.9)^n\ \cos\left(\frac{\pi}{7}\ n\right) u[n]$$

That is, $Y_8(k)$ is the frequency domain sequence obtained by sampling $Y(\omega)$ at $N = 8$ equi-spaced points in the interval $0 \le \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 8-pt inverse DFT of $Y_8(k)$ may be expressed as

$$y_8[n] = \sum_{\ell=-\infty}^{\infty} y[n-\ell 8]\{u[n]-u[n-8]\} \ \underset{8}{\overset{DFT}{\longleftrightarrow}}\ Y_8(k) = \frac{1}{1-0.9e^{j\frac{\pi}{7}}e^{-j2\pi k/8}} + \frac{1}{1-0.9e^{-j\frac{\pi}{7}}e^{-j2\pi k/8}}$$

Determine a simple, closed-form expression for $y_8[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers.

(d) Consider normalizing $y_8[n]$ so that it's first value is one, $\tilde{y}_8[n] = y_8[n]/y_8[0]$, $n = 0, 1, ..., 7$. Compare $\tilde{y}_8[n]$ and $y[n]$ over $n = 0, 1, ..., 7$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

Problem 5. [40 points]

Consider the ARMA(2,2) process generated via the difference equation

$$x[n] = -\frac{1}{4}x[n-2] + \nu[n] + 4\nu[n-2]$$

where $\nu[n]$ is a stationary white noise process with variance $\sigma_\nu^2 = 2$.

(a) Determine a simple, closed-form expression for the autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$ which holds for $-\infty < m < \infty$.

(b) Determine a simple, closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m] = E\{x[n]x[n-m]\}$. Plot $S_{xx}(\omega)$ for $-\pi < \omega < \pi$.

(c) Consider that the power spectrum of the ARMA(2,2) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}^{(1)}}{|1 + a_1(1)e^{-j\omega}|^2}$$

Determine the respective numerical values of the optimum first-order linear prediction coefficient $a_1(1)$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(1)}$.

(d) Consider that the power spectrum of the ARMA(2,2) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}(2)}{|1 + a_2(1)e^{-j\omega} + a_2(2)e^{-j2\omega}|^2}$$

Determine the respective numerical values of the optimum second-order linear prediction coefficients $a_1(2)$ and $a_2(2)$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(2)}$.

Problem 6. [40 points]

We wish to filter a data stream with a filter having the following impulse response

$$h[n] = e^{-j\frac{2\pi}{N}\ell n}\{u[n] - u[n-N]\}$$

where ℓ is an integer between 0 and $N-1$. Convolution with this impulse response may be effected via the following difference equation

$$y[n] = \sum_{k=0}^{N-1} e^{j\frac{-2\pi}{N}\ell k} x[n-k]$$

This implementation requires N multiplications and $N-1$ additions per output point.

(a) It can be shown that we can achieve exactly the same input-output (I/O) relationship via the following difference equation which requires only 1 multiplication and two additions per output point.

$$y[n] = a_1\, y[n-1] + x[n] - x[n-D]$$

Determine the values of a_1 and D in terms of ℓ and N so that this IIR system has exactly the same I/O relationship as the FIR filter above.

(b) Consider the specific case of $\ell = 2$ and $N = 4$:

  (i) State the numerical values of a_1 and D for this case so that the FIR filter and the IIR filter have the same I/O relationship.

  (ii) Plot the pole-zero diagram for the system. Show the region of convergence.

  (iii) Is this a lowpass, bandpass, or highpass filter? Briefly explain why.

  (iv) Plot the impulse response of the system (Stem plot).

(c) Consider the difference equation below:

$$y[n] = -y[n-1] + x[n] - x[n-4]$$

Let the input $x[n]$ be a stationary white noise process with variance $\sigma_x^2 = 1$.

  (i) Determine the autocorrelation sequence $r_{yy}[m] = E\{y[n]y[n-m]\}$. State the numerical values of $r_{yy}[m]$ for $m = 0, 1, 2, 3$ – four answers needed. Is $r_{yy}[m] = 0$ for $m > 3$? Why or why not?

  (ii) Determine the spectral density of $y[n]$, $S_{yy}(\omega)$, the DTFT of $r_{yy}[m] = E\{y[n]y[n-m]\}$. Plot $S_{yy}(\omega)$ for $-\pi < \omega < \pi$.