**Cover Sheet**

Test Duration: 2 hours.
Open Book but Closed Notes.
Calculators allowed (but not necessary).
This test contains **five** problems.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

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Digital Signal Processing I Final Exam 9 Dec. 2002

Given Noble’s identities to use in Problem 1.

(a) If \( E(\omega) \) in Figure 1(b) in terms of \( G(\omega) \) in Figure 1(a) satisfies \( E(\omega) = G(M\omega) \), the I/O relationship of the system in Figure 1(b) is exactly the same as the I/O relationship of the system in Figure 1(a). This result is known as Noble’s First Identity.

\[ \begin{align*} 
M & G(\omega) \\
\downarrow & \quad \downarrow \\
& H(\omega) \\
& \downarrow \\
E(\omega) & \\
\downarrow & \quad \downarrow \\
& F(\omega) \\
& \downarrow \\
& M \\
\end{align*} \]

Figure 1(a).

Figure 1(b).

(b) If \( F(\omega) \) in Figure 2(b) in terms of \( H(\omega) \) in Figure 2(a) satisfies \( F(\omega) = H(M\omega) \), the I/O relationship of the system in Figure 2(b) is exactly the same as the I/O relationship of the system in Figure 2(a). This result is known as Noble’s Second Identity.

Problem 1. [20 points]

(a) Determine the impulse response \( h[n] \) in Figure 3(b) so that the I/O relationship of the system in Figure 3(b) is exactly the same as the I/O relationship of the system in Figure 3(a). Plot the magnitude AND the phase (two separate plots) of the DTFT of \( h[n] \) over \(-\pi < \omega < \pi\). Hint: Analyze the system of Figure 3(a) in the frequency domain using Noble’s First Identity.

(b) Determine the numerical values of the impulse response \( h_{eq}[n] \) in Figure 4(b) so that the I/O relationship of the system in Figure 4(b) is exactly the same as the I/O relationship of the system in Figure 4(a). Hint: Analyze the system of Figure 4(a) in the time domain using Noble’s First Identity.
Problem 2. [20 points]

In the system below, each of the analysis filters, \( h_0[n] \) and \( h_1[n] \), and each of the two synthesis filters, \( f_0[n] \) and \( f_1[n] \), is a causal FIR filter of length 2. The specific values of \( h_0[n] \) and \( h_1[n] \) are indicated. The arrow denotes the value at \( n = 0 \). (See the hints at the bottom of the page.)

\[
\begin{align*}
\text{where } z_i[n] &= x_i[2n], \quad i = 1, 2 \text{ and } y_i[n] = \sum_{\ell=-\infty}^{\infty} z_i[\ell] \delta[n-2\ell], \quad i = 1, 2. \\text{Determine the numerical values of } f_0[n], n = 0, 1 \text{ and } f_1[n], n = 0, 1, \text{ such that } y[n] = 2x[n-1] \text{ for any input sequence } x[n].
\end{align*}
\]

\textbf{Hint 1.} The series combination of a downsampler followed by an upsampler does NOT reduce to an identity transformation – they don’t “cancel” each other.

\textbf{Hint 2.} This problem can be solved either in the time domain or the frequency domain with about equal complexity.
Problem 3. [20 points]

Consider an input sequence, \( x[n] \), of length \( L = 6 \) and an FIR filter with impulse response \( h[n] \) of length \( M = 6 \) as described below.

\[
x[n] = u[n] - u[n-6] = \{1, 1, 1, 1, 1, 1\}
h[n] = u[n] - u[n-6] = \{1, 1, 1, 1, 1, 1\}
\]

We compute an \( N = 8 \)-pt. DFT of each of these two sequences as

\[
\begin{align*}
\text{DFT} & \quad x[n] \leftrightarrow X_8[k] \\
\text{DFT} & \quad h[n] \leftrightarrow H_8[k]
\end{align*}
\]

Next, we point-wise multiply the DFT sequences to form \( Y_8[k] = X_8[k]H_8[k] \), \( k = 0, 1, ..., 7 \). Finally, we compute an \( N = 8 \)-pt. inverse DFT of \( Y_8[k] \) to obtain \( y_P[n] \). Determine the numerical values of \( y_P[n] \) for \( n = 0, 1, 2, 3, 4, 5, 6, 7 \). **You can solve the problem any way you like but briefly explain how you got your answer. Actually computing the DFT’s is NOT the way to solve this problem.**
Problem 4. [20 points]

Consider the ARMA(1,1) process generated via the difference equation

\[ x[n] = -\frac{1}{2}x[n - 1] + w[n] - w[n - 1] \]

where \( w[n] \) is a stationary white noise process with variance \( \sigma_w^2 = 1 \).

(a) Determine the numerical values of \( r_{xx}[0] \), \( r_{xx}[1] \), \( r_{xx}[2] \), where \( r_{xx}[m] \) is the autocorrelation sequence \( r_{xx}[m] = E\{x[n]\cdot x[n-m]\} \). (Note that \( r_{xx}[m] \) is the inverse DTFT of the spectral density \( S_{xx}(\omega) \) asked for in Part (b) below, but there at least three different ways you can solve this part of the problem.)

(b) Determine a simple closed-form expression for the spectral density for \( x[n] \), \( S_{xx}(\omega) \), which may be expressed as the DTFT of \( r_{xx}[m] \):

\[ S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega} \]

(c) Consider the first-order predictor

\[ \hat{x}[n] = -a_1(1)x[n - 1] \]

Determine the numerical value of the optimum predictor coefficient \( a_1(1) \) and the corresponding minimum mean-square error.
Problem 5. [20 points]
Consider the discrete-time complex-valued random process defined for all \( n \):

\[
x[n] = D + A_1 e^{j(\omega_1 n + \Theta_1)} + A_2 e^{j(\omega_2 n + \Theta_2)} + \nu[n]
\]

where the respective frequencies, \( \omega_1 \) and \( \omega_2 \), of the two complex sinewaves are deterministic but unknown constants. The amplitudes, \( A_1 \) and \( A_2 \), and the constant \( D \) are also deterministic but unknown constants. \( \Theta_1 \) and \( \Theta_2 \) are independent random variables with each uniformly distributed over a \( 2\pi \) interval and \( \nu[n] \) is a stationary random process with zero mean and \( r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n - m]\} = \delta[m] \). That is, \( \nu[n] \) forms an i.i.d. sequence with a variance of unity. Note, \( \nu[n] \) is independent of both \( \Theta_1 \) and \( \Theta_2 \) for all \( n \). The values of the autocorrelation sequence for \( x[n] \), \( r_{xx}[m] = E\{x[n]x^*[n - m]\} \), for three different lag values are given below.

\[
r_{xx}[0] = 5, \quad r_{xx}[1] = -1 + j, \quad r_{xx}[2] = 2, \quad r_{xx}[3] = -1 - j
\]

(a) Determine the numerical values of \( \omega_1 \) and \( \omega_2 \). You have to use what you’ve learned during the parametric spectral analysis portion of this course. You will be given no credit if you simply set up a system of equations to solve based on the form of \( r_{xx}[m] = \sum_{i=1}^{P} A_i^2 e^{j\omega_i m} \) and solve this nonlinear system of equations.

(b) Consider a first-order predictor

\[
\hat{x}[n] = -a_1(1)x[n - 1]
\]

Determine the numerical values of the optimum predictor coefficient \( a_1(1) \), and the numerical value of the corresponding minimum mean-square error.

(c) Consider a second-order predictor

\[
\hat{x}[n] = -a_2(1)x[n - 1] - a_2(2)x[n - 2]
\]

Determine the numerical values of the optimum predictor coefficients \( a_2(1) \) and \( a_2(2) \), and the numerical value of the corresponding minimum mean-square error.

(d) Consider a third-order predictor

\[
\hat{x}[n] = -a_3(1)x[n - 1] - a_3(2)x[n - 2] - a_3(3)x[n - 3]
\]

Determine the numerical values of the optimum predictor coefficients \( a_3(1) \), \( a_3(2) \) and \( a_3(3) \), and the numerical value of the corresponding minimum mean-square error.