EE538	**Final Exam**	**Fall 2001**
Digital Signal Processing I		**12 December 2001**

Cover Sheet

Test Duration: 2 hours.
Open Book but Closed Notes.
Calculators **not** allowed.
This test contains **five** problems.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

No.	Topic(s) of Problem	Points
1.	Digital Upsampling	20
2.	Properties and Design of Symmetric (Linear Phase) FIR Filters	20
3.	Sum of Sinewaves Spectral Analysis	20
4.	Spectral Characteristics of ARMA Processes	20
5.	Autoregressive (AR) Spectral Estimation	20

Digital Signal Processing I Final Exam 12 Dec. 2001

Problem 1. [35 points]

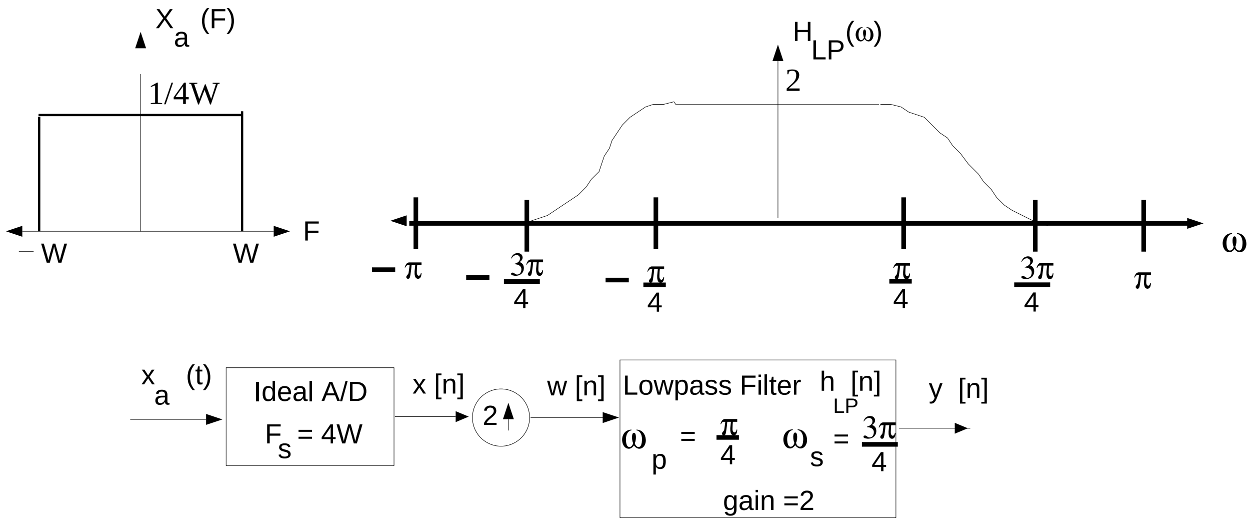

Figure 1.

The analog signal $x_a(t)$ with CTFT $X_a(F)$ shown above is input to the system above, where $x[n] = x_a(n/F_s)$ with $F_s = 4W$, and

$$h_{LP}[n] = \frac{\sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n}\frac{\cos(\frac{\pi}{4}n)}{1-\frac{n^2}{4}}, \qquad -\infty < n < \infty,$$

such that $H_{LP}(\omega) = 2$ for $|\omega| \leq \frac{\pi}{4}$, $H_{LP}(\omega) = 0$ for $\frac{3\pi}{4} \leq |\omega| \leq \pi$, and $H_{LP}(\omega)$ has a cosine roll-off from 1 at $\omega_p = \frac{\pi}{4}$ to 0 at $\omega_s = \frac{3\pi}{4}$. Finally, the zero inserts may be mathematically described as

$$w[n] = \begin{cases} x(\frac{n}{2}), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

(a) Plot the magnitude of the DTFT of the output $y[n]$, $Y(\omega)$, over $-\pi < \omega < \pi$.

(b) Given that

$$x[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n} \qquad -\infty < n < \infty,$$

provide an analytical expression for $y[n]$ for $-\infty < n < \infty$ (similar to the expression for either $x[n]$ above, for example.)

(c) The up-sampling by a factor of 2 in Figure 1 can be efficiently done via the block diagram in Figure 2 on the next page. **THIS PROBLEM IS CONTINUED ON THE NEXT PAGE.**

(i) Provide an analytical expression for $h_0[n] = h_{LP}[2n]$ for $-\infty < n < \infty$. Simplify as much as possible.

(ii) Plot the magnitude of the DTFT of $h_0[n]$, $|H_0(\omega)|$, over $-\pi < \omega < \pi$.

(iii) Provide an analytical expression for the output $y_0[n]$ for $-\infty < n < \infty$. Is $y_0[n] = x[n]$? Explain why they are the same if you said "YES" or explain why they are not the same if you said "NO."

(iv) Describe an advantage of employing this lowpass filter, $h_{LP}[n]$, in the process of upsampling by a factor of 2.

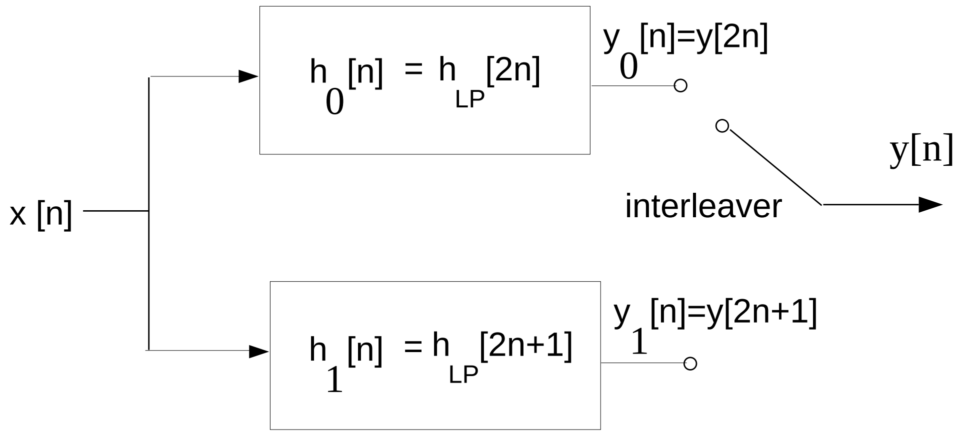

Figure 2.

Problem 2. [20 points]

Let $h[n], n = 0, 1, 2$, be the impulse response of a symmetric FIR filter of length $M = 3$. The frequency response of the filter is the DTFT

$$H(\omega) = \sum_{n=0}^{2} h[n]e^{-j\omega n}$$

Suppose we desire to design a LPF with passband edge, $\omega_p = \pi/6$. The design criterion for selecting the filter coefficients, {h[0], h[1], h[2]}, is to maximize the ratio of the energy in the passband to the total energy, i.e.,

$$\underset{\{h[0],h[1],h[2]\}}{Maximize} \quad \frac{\frac{1}{2\pi}\int_{-\omega_p}^{\omega_p} |H(\omega)|^2 d\omega}{\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega}$$

where $\omega_p = \pi/6$. Determine the specific numerical values of {h[0], h[1], h[2]} that meet this design criterion, i. e., solve the above optimization problem. In fact, since the filter is symmetric $h[2] = h[0]$ and we can also assign $h[1] = 1$ without loss of generality. Constraining the problem in this manner, the primary task is to find the value of $h[0]$. Clearly indicate the steps required in arriving at the solution and show all work. Note that $\sin(\pi/6) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$. If you lose the latter sine value, you can simply carry the $\sqrt{3}$ throughout the calculation. You don't need to approximate it by a numerical value.

Problem 3. [20 points]

Consider the discrete-time complex-valued random process defined for all n:

$$x[n] = D + A_1 e^{\jmath(\omega_1 n + \Theta_1)} + A_2 e^{\jmath(\omega_2 n + \Theta_2)} + \nu[n]$$

where the respective frequencies, ω_1 and ω_2, of the two complex sinewaves are deterministic but unknown constants. The amplitudes, A_1 and A_2, and the constant D are also deterministic but unknown constants. Θ_1 and Θ_2 are independent random variables with each uniformly distributed over a 2π interval and $\nu[n]$ is a stationary random process with zero mean and $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \delta[m]$. That is, $\nu[n]$ forms an i.i.d. sequence with a variance of unity. Note, $\nu[n]$ is independent of both Θ_1 and Θ_2 for all n. The values of the autocorrelation sequence for $x[n]$, $r_{xx}[m] = E\{x[n]x^*[n-m]\}$, for three different lag values are given below.

$$r_{xx}[0] = 5, \quad r_{xx}[1] = -1 + j, \quad r_{xx}[2] = 2, \quad r_{xx}[3] = -1 - j$$

(a) Determine the numerical values of ω_1 and ω_2. **You have to use what you've learned during the parametric spectral analysis portion of this course. You will be given no credit if you simply set up a system of equations to solve based on the form of $r_{xx}[m] = \sum_{i=1}^{p} A_i^2 e^{j\omega_i m}$ and solve this nonlinear system of equations.** You might want to do the other parts of this problem first.

(b) Consider a first-order predictor

$$\hat{x}[n] = -a_1(1)x[n-1]$$

Determine the numerical values of the optimum predictor coefficient $a_2(1)$, and the numerical value of the corresponding minimum mean-square error.

(c) Consider a second-order predictor

$$\hat{x}[n] = -a_2(1)x[n-1] - a_2(2)x[n-2]$$

Determine the numerical values of the optimum predictor coefficients $a_2(1)$ and $a_2(2)$, and the numerical value of the corresponding minimum mean-square error.

(d) Consider a second-order predictor

$$\hat{x}[n] = -a_3(1)x[n-1] - a_3(2)x[n-2] - a_3(3)x[n-3]$$

Determine the numerical values of the optimum predictor coefficients $a_2(1)$ and $a_2(2)$, and the numerical value of the corresponding minimum mean-square error.

Problem 4. [20 points]

Consider the ARMA(1,1) process generated via the difference equation

$$x[n] = \frac{1}{4}x[n-1] + w[n] - w[n-1]$$

where $w[n]$ is a stationary white noise process with variance $\sigma_w^2 = 1$.

(a) Determine the numerical values of $r_{xx}[0]$, $r_{xx}[1]$, $r_{xx}[2]$, where $r_{xx}[m]$ is the autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$. (Note that $r_{xx}[m]$ is the inverse DTFT of the spectral density $S_{xx}(\omega)$ asked for in Part (b) below, but there at least three different ways you can solve this part of the problem.)

(b) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega}$$

(c) Consider the first-order predictor

$$\hat{x}[n] = -a_1(1)x[n-1]$$

Determine the numerical value of the optimum predictor coefficient $a_1(1)$ and the corresponding minimum mean-square error.

Problem 5. [20 points]

Suppose that the random process $x[n]$ is the output of a stable LTI system with impulse response

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 2^n & n < 0 \end{cases}$$

when the input $\nu[n]$ is a zero-mean white noise process with variance σ^2. In the following let $H(z)$ denote the Z-Transform of $h[n]$ and let

$$\hat{x}_p[n] = -\sum_{k=1}^{p} a_p[k]x[n-k]$$

denote the order p minimum mean–square linear predictor of $x[n]$ given $\{x[n-k] : 1 \leq k \leq p\}$. Let $f_p[n] = x[n] - \hat{x}_p[n]$ be the prediction error, let $E_p = \mathbf{E}\{|f_p[n]|^2\}$, and let

$$A_p(z) = 1 + \sum_{k=1}^{p} a_p[k]z^{-k}$$

denote the order p prediction error filter.

(a) Find the transfer function $H(z)$ of the system and indicate its region of convergence.

(b) Find the (true) power spectral density of $x[n]$, $S_{xx}(\omega)$.

(c) Suppose that another LTI system is placed in series with $H(z)$ having a transfer function $P(z)$. The new output is called $y[n]$. If $P(z)$ is an all-pass filter for which $|P(\omega)| = 1$ for all ω, find the (true) power spectral density of $y[n]$, $S_{yy}(\omega)$. Is $r_{yy}[m]$ equal to $r_{xx}[m]$? Explain your answer.

(d) For the original system $H(z)$ and the process $x[n]$ determine the cofficients, $a_2[1]$ and $a_2[2]$, of the optimum second order linear predictor. *Hint:* A first-order all-pass filter has a transfer function of the form

$$H_{all-pass}(z) = -a\,\frac{z - \frac{1}{a}}{z - a}$$