EE538 DSP I inal Exam Sol'n. F'99

Solution to Prob. 1

$$y(t) = x(t) * g(t)$$

$$= x(t) * \{\delta(t) + \delta(t-\tau)\}$$

$$= x(t) + x(t-\tau)$$

$$= \sum_{k=-\infty}^{\infty} b[k] \{p(t-kT_0) + p(t-kT_0-\tau)\}$$

$$y[n] = \sum_{k=-\infty}^{\infty} b[k] \{p(nT_0 - kT_0) + p(\underbrace{nT_0 - kT_0 - \tau}_{(n-k-1)T_0})\}$$

for $\tau = T_0$

$$= \sum_{k=-\infty}^{\infty} b[k] \{p[n-k] + p[n-k-1]\}$$

$$= \sum_{k=-\infty}^{\infty} b[k]\, h[n-k]$$

where: $h[n] = p[n] + p[n-1]$

$$p[n] = p(nT_0) = \tilde{p}[2n] = 4\delta[n]$$

Answer to (a) $\Big\{$ Thus: $h[n] = 4\delta[n] + 4\delta[n-1]$

Sol'n. to Prob. 1 (cont.)

for $\tau = \frac{T_0}{2}$:

$$y[n] = \sum_{k=-\infty}^{\infty} b[k]\left\{ p(nT_0 - kT_0) + p\left(nT_0 - kT_0 - \frac{T_0}{2}\right)\right\}$$

$$= \sum_{k=-\infty}^{\infty} b[k]\left\{ p\left(2(n-k)\frac{T_0}{2}\right) + p\left(\left[2(n-k)-1\right]\frac{T_0}{2}\right.\right.$$

$$= \sum_{k=-\infty}^{\infty} b[k]\, h[n-k]$$

where: $h[n] = \tilde{p}[2n] + \tilde{p}[2n-1]$

$$= 4\delta[n] + \{1, \underset{\substack{\uparrow \\ n=0}}{-2}, -2, 1\}$$

$$= \{1, \underset{\substack{\uparrow \\ n=0}}{2}, -2, 1\}$$

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Sol'n. to Prob. 2

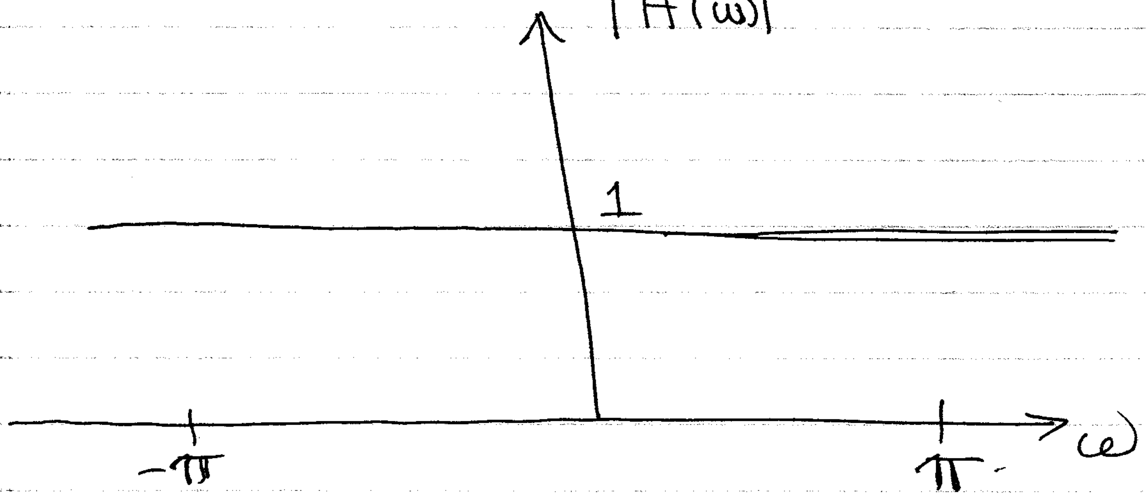
(a) is below.

(b) $$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{2}\,\frac{e^{j\omega}-2}{e^{j\omega}-\frac{1}{2}}$$

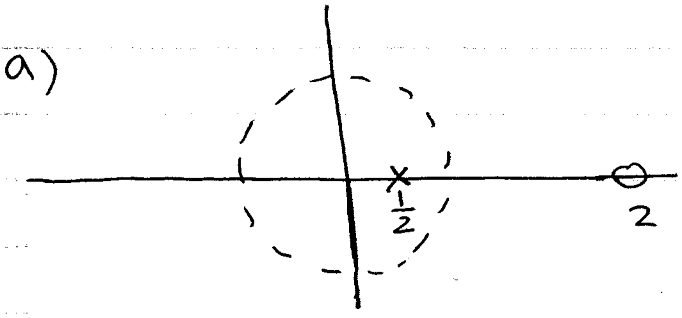
$$|H(\omega)|^2 = \frac{1}{4}\,\frac{e^{j\omega}-2}{e^{j\omega}-\frac{1}{2}} \cdot \frac{e^{-j\omega}-2}{e^{-j\omega}-\frac{1}{2}}$$

$$= \frac{1-4\cos\omega\quad+4}{4\left(1-\cos\omega+\frac{1}{4}\right)} = \frac{5-4\cos\omega}{5-4\cos\omega}$$

$$= 1 \quad \forall \quad \omega$$



(a)



ROC:

$\frac{1}{2} < |z| < \infty$

$\Rightarrow |z| > 1/2$

Signal Processing

Sol'n. to Problem 3

$$\frac{1}{2\pi}\int_{-\omega_p}^{\omega_p}\left|\sum_{n=0}^{2} h(n)e^{-j\omega n}\right|^2 d\omega = \sum_{n=0}^{2}\sum_{m=0}^{2} h(n)h(m)\underbrace{\frac{1}{2\pi}\int_{-\omega_p}^{\omega_p} e^{j(m-n)\omega}\,du}_{\frac{\sin[(m-n)\omega_p]}{\pi(m-n)}}$$

$$= \frac{\underline{h}^T \underline{M}\,\underline{h}}{\underline{h}^T \underline{h}} \quad \Rightarrow \underline{h} = [h(0), h(1), h(2)]^T$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(\omega)|^2 d\omega = \sum_{n=0}^{2} h^2(n) = \underline{h}^T\underline{h}$$

- note: $\frac{\sin[(m-n)\frac{\pi}{2}]}{\pi(m-n)} \Rightarrow m, n = 0, 1, 2$

$$\Rightarrow \quad \underline{M} = \frac{1}{\pi}\begin{bmatrix} \pi/2 & 1 & 0 \\ 1 & \pi/2 & 1 \\ 0 & 1 & \pi/2 \end{bmatrix}$$

- Minimize $\underset{\underline{h}}{\ }$ $\frac{\underline{h}^T\underline{M}\,\underline{h}}{\underline{h}^T\underline{h}}$

$\Rightarrow$ sol'n.: $\underline{h}$ is that eigenvector of $\underline{M}$ associated with largest eigenvalue

- compute eigenvalues of $\underline{M}$

$$\begin{vmatrix} \lambda-\frac{1}{2} & -\frac{1}{\pi} & 0 \\ -1/\pi & \lambda-\frac{1}{2} & -1/\pi \\ 0 & -1/\pi & \lambda-\frac{1}{2} \end{vmatrix} = (\lambda-1)\left\{(\lambda-\tfrac{1}{2})^2 - \frac{1}{\pi^2}\right\} - \frac{1}{\pi^2}(\lambda-\tfrac{1}{2})$$

$$= (\lambda-\tfrac{1}{2})\left\{(\lambda-\tfrac{1}{2})^2 - \frac{1}{\pi^2} - \frac{1}{\pi^2}\right\}$$

Signal Processing

Sol'n. to Prob 3 (cont.)

$$= \left(\lambda - \frac{1}{2}\right) \underbrace{\left\{ \lambda^2 - \lambda + \frac{1}{4} - \frac{2}{\pi^2} \right\}}_{}$$

$$\text{roots} = \frac{1 \pm \sqrt{1 - 4\left(\frac{1}{4} - \frac{2}{\pi^2}\right)}}{2}$$

- largest eigenvalue $= \frac{1}{2} + \frac{\sqrt{2}}{\pi}$

- largest eigenvector:

$$\begin{bmatrix} \frac{\sqrt{2}}{\pi} & -\frac{1}{\pi} & 0 \\ -\frac{1}{\pi} & \frac{\sqrt{2}}{\pi} & -\frac{1}{\pi} \\ 0 & -\frac{1}{\pi} & \frac{\sqrt{2}}{\pi} \end{bmatrix} \begin{bmatrix} 1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-\frac{1}{\pi} e_2 = -\frac{\sqrt{2}}{\pi} \Rightarrow e_2 = \sqrt{2}$$

$$-\frac{1}{\pi}\sqrt{2} + \frac{\sqrt{2}}{\pi} e_3 = 0 \Rightarrow e_3 = 1$$

$$h_{opt} = [\underset{\overset{\uparrow}{h(0)}}{1}, \underset{\overset{\uparrow}{h(1)}}{\sqrt{2}}, \underset{\overset{\uparrow}{h(2)}}{1}]$$

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Sol'n. to Prob. (4)

$$(a)\ w[n] = e^{j\frac{\pi}{M}n}\left\{u[n] - u[n-\tfrac{M}{2}]\right\}$$

$$* e^{-j\frac{\pi}{M}n}\left\{u[n]-u[n-\tfrac{M}{2}]\right\}$$

- for $n = 0, 1, \ldots, \frac{M}{2}-1$:

$$w[n] = \sum_{k=0}^{n} e^{j\frac{\pi}{M}k} e^{-j\frac{\pi}{M}(n-k)}$$

$$= e^{-j\frac{\pi}{M}n} \sum_{k=0}^{n} e^{j\frac{2\pi}{M}k}$$

$$= e^{-j\frac{\pi}{M}n} \frac{1-e^{j\frac{2\pi}{M}(n+1)}}{1-e^{j\frac{2\pi}{M}}}$$

$$= e^{-j\frac{\pi}{M}n} \frac{e^{j\frac{\pi}{M}(n+1)}}{e^{j\frac{\pi}{M}}} = \frac{\sin\left(\frac{\pi}{M}(n+1)\right)}{\sin\left(\frac{\pi}{M}\right)}$$

$$w[n] = \frac{\sin\left[\frac{\pi}{M}(n+1)\right]}{\sin\left[\frac{\pi}{M}\right]} \quad \text{for } n = 0, 1, \ldots, \frac{M}{2}-1$$

- for $n = \frac{M}{2}, \ldots, M-2$:

$$w[n] = \sum_{k=n-\frac{M}{2}+1}^{M/2-1} e^{j\frac{\pi}{M}k} e^{-j\frac{\pi}{M}(n-k)}$$

Sol'n. to Prob. 4 (cont.)

- change of variables: $k' = k - \left(n - \frac{M}{2} + 1\right) = k - n + \frac{M}{2} - 1$ $\Big\}\ k = k' + n - \frac{M}{2} + 1$

  - limits: $k'\Big|_{0}^{\frac{M}{2} - 1 - n + \frac{M}{2} - 1} = M - n - 2$

$$w[n] = \sum_{k'=0}^{M-2-n} e^{j\frac{2\pi}{M}\left(k' + n - \frac{M}{2} + 1\right)} \cdot e^{-j\frac{\pi}{M}n}$$

$$= e^{-j\frac{\pi}{M}n} e^{j\frac{2\pi}{M}n} e^{-j\pi} e^{j\frac{2\pi}{M}} \sum_{k=0}^{M-2-n} e^{j\frac{2\pi}{M}k}$$

$$= e^{j\frac{\pi}{M}n} e^{j\frac{2\pi}{M}} (-1) \frac{1 - e^{j\frac{2\pi}{M}(M-1-n)}}{1 - e^{j\frac{2\pi}{M}}}$$

$$= e^{j\frac{\pi}{M}n} e^{j\frac{2\pi}{M}} \frac{e^{-j\frac{\pi}{M}(n+1)}}{e^{j\frac{\pi}{M}}} \frac{\sin\left(\frac{\pi}{M}(n+1)\right)}{\sin\left(\frac{\pi}{M}\right)}$$

* ANSWER 2(a) * *

$$= \frac{\sin\left(\frac{\pi}{M}(n+1)\right)}{\sin\left(\frac{\pi}{M}\right)} \quad \text{for} \quad n = \frac{M}{2}, \frac{M}{2}+1, \ldots, M-2$$

- from before, for $n = 0, 1, \ldots, \frac{M}{2} - 1$:

$$w[n] = \frac{\sin\left(\frac{\pi}{M}(n+1)\right)}{\sin\left(\frac{\pi}{M}\right)}$$

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Sol'n. to Prob. 4

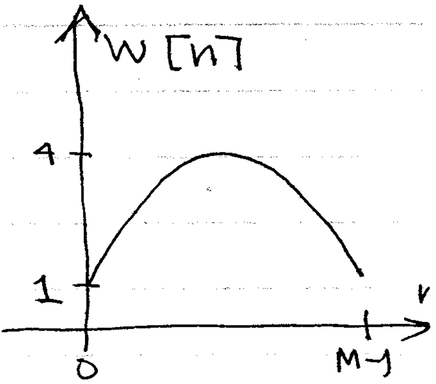
(b) $w[n] = w[M-2-n]$ $\Big\}$ Symmetric!

$n = 0, 1, \ldots, M-2$

Proof:

$$\frac{\sin\left[\frac{\pi}{M}(M-2-n+1)\right]}{\sin\left(\frac{\pi}{M}\right)} = \frac{\sin\left[\pi - \frac{\pi}{M}(n+1)\right]}{\sin\left(\frac{\pi}{M}\right)}$$

$$= +\frac{\sin\left[\frac{\pi}{M}(n+1)\right]}{\sin\left(\frac{\pi}{M}\right)}$$



essentially one-half cycle of a sinewave

(c) See plot attached.

$$W_r(\omega) = \frac{\sin\left[\frac{M}{4}\left(\omega - \frac{\pi}{M}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega - \frac{\pi}{M}\right)\right]} \cdot \frac{\sin\left[\frac{M}{4}\left(\omega + \frac{\pi}{M}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega + \frac{\pi}{M}\right)\right]}$$

nulls at $\omega = \frac{\pi}{M} \pm \ell \frac{4\pi}{M}$, $\ell = 0, 1, \ldots, \frac{M}{4} - 1$

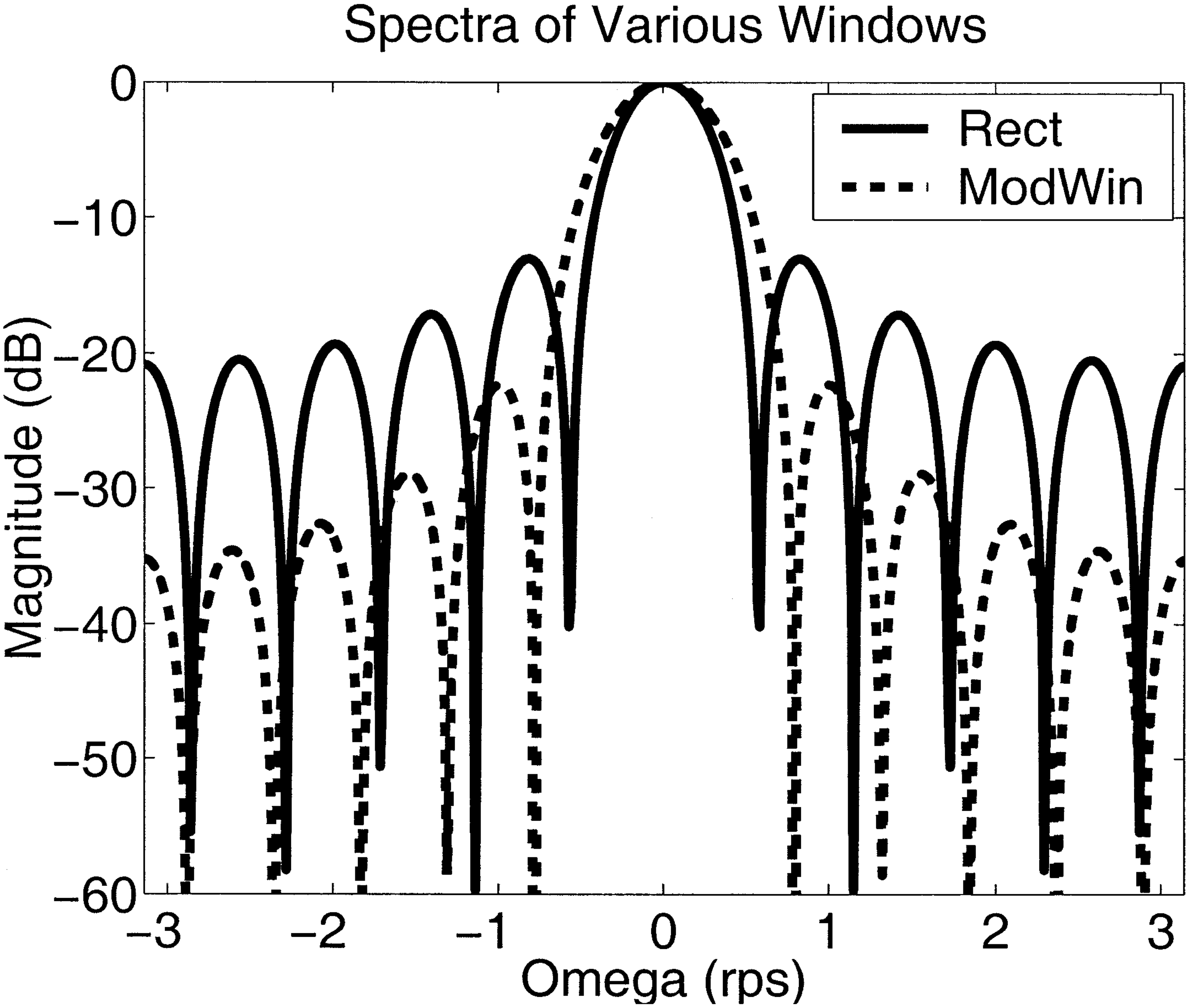
and at $= -\frac{\pi}{M} \pm \ell \frac{4\pi}{M}$, " "

(d) mainlobe width: $\frac{6\pi}{M}$ (null-to-null)

This is larger than the mainlobe width for a rectangular window of length M-1 which is $\frac{4\pi}{M-1} \approx \frac{4\pi}{M}$

Plots of $|W(\omega)|$ and $|W_{RECT}(\omega)|$ for $M=16$.

Both $w[n]$ and $w_{RECT}[n]$ are of length $M-1=15$



(e) peak sidelobe is lower

(f) sidelobes are lower, in general

} due to tapering see plot of $w[n]$

Final Exam

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- $r_{xx}[m] = r_{ss}[m] + \delta[m]$ where:

$r_{ss}[m] = E\{s[n]\,s[n-m]\}$, $s[n] = A\cos(\omega_0 n + \Theta)$

$\Rightarrow r_{ss}[0] = r_{xx}[0] - 1$; $r_{xx}[m] = r_{ss}[m]$, $m = 1, 2$

Given: $r_{ss}[0] = 2$; $r_{ss}[1] = 1$; $r_{ss}[2] = -1$

(a)

$$\begin{bmatrix} r_{ss}[0] & r_{ss}[1] \\ r_{ss}[1] & r_{ss}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} r_{ss}[1] \\ r_{ss}[2] \end{bmatrix}$$

Ⓐ $2a_1 + a_2 = -1$

Ⓑ $a_1 + 2a_2 = 1$

-2Ⓐ $+$Ⓑ $\Rightarrow (-4+1)a_1 = 2+1 = 3$

$a_1 = -1$

$a_2 = -1 - 2a_1 = -1 - 2(-1) = 1$

$\left.\begin{matrix} \\ \end{matrix}\right\}$ $a_1 = -1$, $a_2 = 1$

(b) $r_{ss}[3] = -a_1\, r_{ss}[2] - a_2\, r_{ss}[1]$

$r_{ss}[3] = r_{ss}[2] - r_{ss}[1]$

$= -1 - 1 = -2$

(c) $S_{xx}(\omega) = ?$ $r_{xx}[m] = \left(\frac{A}{\sqrt{2}}\right)^2 \cos(\omega_0 m) + \delta[m$

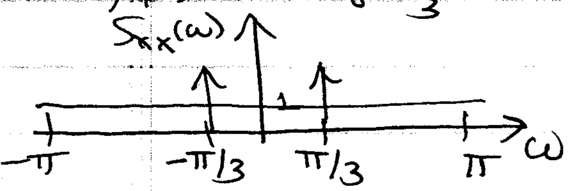
$$S_{xx}(\omega) = \frac{A^2}{2}\left\{\pi\delta(\omega-\omega_0) + \pi\delta(\omega-\omega_0)\right\}$$

$\omega_0 = ?$ $\Rightarrow$ $z^2 + a_1 z + a_2 = z^2 - z + 1$

$= (z - e^{j\omega_0})(z - e^{-j\omega_0})$

$= (z - e^{j\frac{\pi}{3}})(z - e^{-j\frac{\pi}{3}})$

Answer: $\omega_0 = \frac{\pi}{3}$



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Sol'n. to Prob. 6 $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$

(a) $$H(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2}$$

ROC: $\frac{1}{2} < |z| < 2$

stable implies ROC includes $|z|=1$

$$H(z) = \frac{z\left(z-2-\left(z-\frac{1}{2}\right)\right)}{\left(z-\frac{1}{2}\right)(z-2)}$$

$$= \frac{-\frac{3}{2}z}{\left(z-\frac{1}{2}\right)(z-2)}$$

(b)

$$S_{xx}(\omega) = |H(\omega)|^2 S_{vv}(\omega)$$

$$= \frac{\frac{9}{4}\sigma^2}{\left|e^{j\omega}-\frac{1}{2}\right|^2 \left|e^{j\omega}-2\right|^2}$$

(c) Since $P(\omega)$ is all-pass, $|P(\omega)|=1$

and $S_{yy}(\omega) = |P(\omega)|^2 S_{xx}(\omega)$

$$= S_{xx}(\omega)$$

thus: $r_{yy}[m] = r_{xx}[m]$

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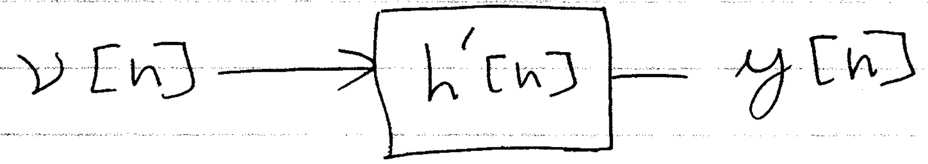
Sol'n. to Prob. 6

(d) Choose $P(z)$ from Prob. 2



$$H'(z) = \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)} \frac{(z-2)}{(z-\frac{1}{2})} = \frac{-\frac{3}{2}z}{(z-\frac{1}{2})^2}$$

$$= \frac{-\frac{3}{2}z}{z^2 - z + \frac{1}{4}} = \frac{-\frac{3}{2}z}{z^2 + a_1 z + a_2}$$

$y[n]$ is an AR(2) process (stable)
(and $S_{yy}(\omega) = S_{xx}(\omega)$, $r_{yy}[m] = r_{xx}[m]$)

· Thus, best 2nd-order linear predictor are the AR model parameters

$$a_2[1] = a_1 = -1$$
$$a_2[2] = a_2 = \frac{1}{4}$$