Problem 1

(a) $y(n) = x(n) - x(n-1) + \frac{1}{2}x(n-2)$

(b) $y(n) = x(n) + K_1 x(n-1) + K_2\{K_1 x(n-1) + x(n-2)$

$= x(n) + K_1(K_2+1)x(n-1) + K_2 x(n-2)$

(c) recall, answer to part (a):

$y(n) = x(n) - x(n-1) + \frac{1}{2}x(n-2)$

· comparing: $K_2 = \frac{1}{2}$ $\quad K_1(K_2+1) = -1$

$K_1 = -\frac{2}{3}$

Sol'n. to Prob. 2

$$x(n) = x_a\left(\frac{n}{8}\right) = e^{-(4\ln 2)\frac{1}{8}n}\cos\left(2\pi\frac{n}{8}\right)u(n)$$

$$= \left(e^{\ln 2}\right)^{-\frac{n}{2}}\cos\left(\frac{\pi}{4}n\right)u(n)$$

$$= \left(2^{-\frac{1}{2}}\right)^n\cos\left(\frac{\pi}{4}n\right)u(n) =$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{\pi}{4}n\right)u(n)$$

(a) Table 4.3 of Z-Transform pairs entry:

$$a^n\cos\omega_0 n\, u(n) \xleftrightarrow{ZT} \frac{1 - a\cos\omega_0 z^{-1}}{1 - 2a\cos\omega_0 z^{-1} + a^2 z^{-2}}$$

ROC: $|z| > |a|$

Sol'n. to Prob. 2 (cont.)

• Thus:

$$\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right)u(n) \xleftrightarrow{ZT} \frac{1-\frac{1}{\sqrt{2}}\cos\frac{\pi}{4}\, z^{-1}}{1-2\left(\frac{1}{\sqrt{2}}\right)\cos\frac{\pi}{4}\, z^{-1}+\frac{1}{2}z^{-2}}$$

$$\boxed{\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}} \qquad X(z)=\frac{z\left(z-\frac{1}{2}\right)}{z^2-z+\frac{1}{2}}$$

Region of convergence:

$$|z|>\frac{1}{\sqrt{2}}$$

(b)

$$X(\omega)=X(z)\Big|_{z=e^{j\omega}}=\frac{e^{j\omega}\left(e^{j\omega}-\frac{1}{2}\right)}{\left(e^{j\omega}-\left(\frac{1+j}{2}\right)\right)\left(e^{j\omega}-\left(\frac{1-j}{2}\right)\right)}$$

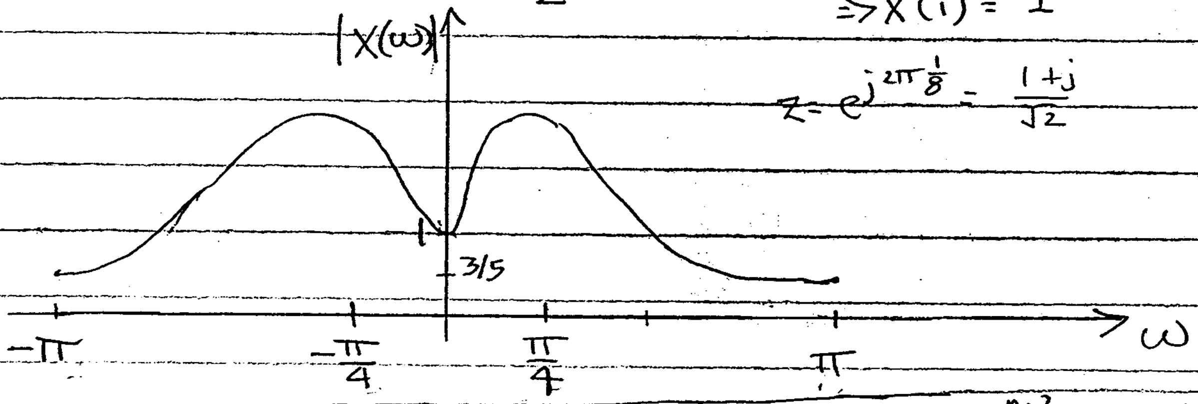
• zero at $z=1/2$

• poles at $p_1=\frac{1+j}{2}$, $p_2=\frac{1-j}{2}$

$z=1 \Rightarrow e^{j2\pi(0)}$

$\Rightarrow X(1)=1$

$z=e^{j2\pi\frac{1}{8}}=\frac{1+j}{\sqrt{2}}$



Ranjit: $y(n)=\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{\pi}{4}n\right)u(n)-\left(\frac{1}{\sqrt{2}}\right)^{n-1}\cos\left(\frac{\pi}{4}(n-1)\right)u(n-1)+\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{n-2}\cos\left(\frac{\pi}{4}(n-2)\right)u(n-2)$

acceptable but R-3 pts. for not simplifying

(c) $Y(z)=H(z)X(z)$

$$=\frac{z^2-z+1/2}{z^2}\cdot\frac{z\left(z-\frac{1}{2}\right)}{z^2-z+1/2}$$

$$=\frac{z-1/2}{z}=1-\frac{1}{2}z^{-1}$$

Solution to Problem 3:

Since $x(n)$ and $h(n)$ are both of length 6, their lin
convolution is of length $6+6-1=11$

⇒ Thus, both $y_6(n)$ and $y_9(n)$ are then corrupted by aliasing.

⇒ let $y(n)$, $n=0,1,\ldots,10$ denote the linear convol of $x(n)$ and $h(n)$

⇒ From derivations done in class, we have

(1) $$y_6(n) = \sum_{\ell=-\infty}^{\infty} y(n+6\ell) = y(n)+y(n+6) \quad \text{for } n=0,1,2,3,4,5$$

(2) $$y_9(n) = \sum_{\ell=-\infty}^{\infty} y(n+9\ell) = y(n)+y(n+9) \quad \text{for } n=0,1,2,\ldots,7,8$$

From (1), we have that

$$\left.\begin{aligned} y_6(0) &= y(0)+y(6) = 21 \\ y_6(1) &= y(1)+y(7) = 21 \\ y_6(2) &= y(2)+y(8) = 21 \\ y_6(3) &= y(3)+y(9) = 21 \\ y_6(4) &= y(4)+y(10) = 21 \end{aligned}\right\} \text{5 pts. corrupted by aliasing}$$

$$y_6(5) = y(5) = 21 \quad \} \text{ 1 pt. OK}$$

⇒ We can immediately say $y(5)=21$

From (2), we have:

$y_9(0) = y(0) + y(9) = 9$
$y_9(1) = y(1) + y(10) = 12$
2 pts. corrupted by aliasing

$y_9(2) = y(2) = 15$
$y_9(3) = y(3) = 18$
$y_9(4) = y(4) = 20$
$y_9(5) = y(5) = 21$
$y_9(6) = y(6) = 15$
$y_9(7) = y(7) = 10$
$y_9(8) = y(8) = 6$
7 pts. OK

Thus, all we need to find is $y(0), y(1), y(9), y$

Since $y(6) = 15$, $y(0) = y_6(0) - y(6) = 21 - 15 = 6$

Since $y(7) = 10$, $y(1) = y_6(1) - y(7) = 21 - 10 = 1$

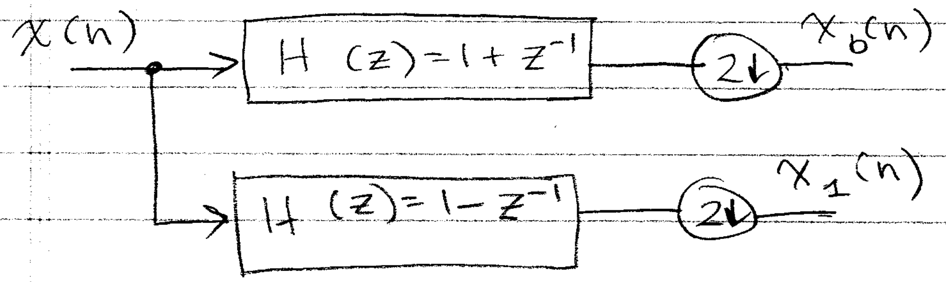
Since $y(0) = 6$, $y(9) = y_9(0) - y(0) = 9 - 6 = 3$

Since $y(1) = 11$, $y(10) = y_9(1) - y(1) = 12 - 11 =$

Answer:

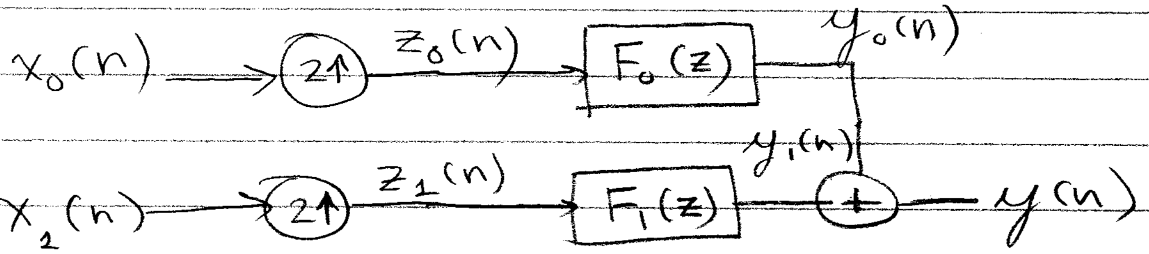
n	0	1	2	3	4	5	6	7	8	9	10
y(n)	6	11	15	18	20	21	15	10	6	3	1

Solution to Problem 4



$$X_i(z) = \frac{1}{2} H_i(z^{\frac{1}{2}}) X(z^{\frac{1}{2}}) + \frac{1}{2} H_i(-z^{\frac{1}{2}}) X(-z^{\frac{1}{2}})$$

$i = 0, 1$



$$Z_i(z) = X_i(z^2)$$

$$= \frac{1}{2} H_i(z) X(z) + \frac{1}{2} H_i(-z) X(-z)$$

$i = 0, 1$

$$Y(z) = \frac{1}{2}\left\{ H_0(z) F_0(z) + H_1(z) F_1(z) \right\} X(z)$$

$$+ \frac{1}{2}\left\{ H_0(-z) F_0(z) + H_1(-z) F_1(z) \right\} X(-z)$$

for $y(n) = x(n-1) \Rightarrow Y(z) = z^{-1} X(z)$, require:

$$\frac{1}{2}(1 + z^{-1}) F_0(z) + \frac{1}{2}(1 - z^{-1}) F_1(z) = z^{-1}$$

$$\frac{1}{2}(1 - z^{-1}) F_0(z) + (1 + z^{-1}) F_1(z) = 0$$

Sol'n. to Prob. 4 (cont.)

In matrix form:

$$\underbrace{\begin{bmatrix} 1+z^{-1} & 1-z^{-1} \\ 1-z^{-1} & 1+z^{-1} \end{bmatrix}}_{\text{inverse:}} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \begin{bmatrix} 2z^{-1} \\ 0 \end{bmatrix}$$

inverse:

$$\frac{1}{(1+z^{-1})^2-(1-z^{-1})^2} \begin{bmatrix} 1+z^{-1} & -1+z^{-1} \\ -1+z^{-1} & 1+z^{-1} \end{bmatrix}$$

- Thus:

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \frac{1}{4z^{-1}} \begin{bmatrix} 1+z^{-1} & -1+z^{-1} \\ -1+z^{-1} & 1+z^{-1} \end{bmatrix} \begin{bmatrix} 2z^{-1} \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} F_0(z) &= \tfrac{1}{2}(1+z^{-1}) \\ F_1(z) &= \tfrac{1}{2}(-1+z^{-1}) \end{aligned}\right\} \text{Answers}$$

~~Solution to Problem 6~~

(a) from text pg. 605

$$h_d(n) = \frac{\cos\left(\pi\left(n - \frac{M-1}{2}\right)\right)}{n - \frac{M-1}{2}} = \frac{(-1)^{n-\frac{M-1}{2}}}{n - \frac{M-1}{2}}$$

(b) $w_1(n) = -w_1(M-1-n) \Rightarrow$ anti-symmetric

$w_2(n) = w_2(M-1-n) \Rightarrow$ symmetric

since $h_d(n) = -h_d(M-1-n)$ and require the product $h(n) = h_d(n)w(n)$ to be anti-symmetric (to get purely imaginary frequency response), must choose $w(n) = w_2(n)$ as product of anti-symmetric $h_d(n)$ with symmetric $w_2(n)$ yields anti-symmetric $h(n)$

Solution to Problem 5

(a)
$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{8} \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$-2\,(1) + (2) \Rightarrow \quad -\frac{3}{2}a_1 = 1 - \frac{1}{8} = \frac{7}{8}$

$$a_1 = -\frac{2}{3}\cdot\frac{7}{8} = -\frac{7}{12}$$

$$a_2 = -\frac{1}{2}\left(\frac{-7}{12}\right) - \frac{1}{8} = \frac{7}{24} - \frac{3}{24} = \frac{4}{24} = \frac{1}{6}$$

$$a_1 = \frac{-7}{12} \qquad a_2 = \frac{1}{6}$$

$$\sigma_w^2 = r_{xx}(0) + a_1 r_{xx}(1) + a_2 r_{xx}(2)$$
$$= 1 - \frac{7}{12}\left(\frac{1}{2}\right) + \frac{1}{6}\left(\frac{1}{8}\right) = \frac{48 - 14 + 1}{48} = \frac{35}{48}$$

Solution to Problem **5** (cont.)

$$r_{xx}(3) = -a_1 r_{xx}(2) - a_2 r_{xx}(1)$$

$$= +\frac{7}{12}\left(+\frac{1}{8}\right) - \frac{1}{6}\left(\frac{1}{2}\right) =$$

$$= \frac{+7-8}{96} = -\frac{1}{96}$$

$$r_{xx}(4) = -a_1 r_{xx}(3) - a_2 r_{xx}(2)$$

$$= \frac{7}{12} \cdot \frac{1}{96} - \frac{1}{6}\frac{1}{8}$$

$$= \frac{7 - 2(12)}{12 \cdot 96} = \frac{-31}{12 \cdot 96} = \frac{-31}{1152}$$

(c) $$P_{xx}(f) = \frac{\sigma_u^2}{\left|1 + a_1 e^{-j2\pi f} + a_2 e^{-j4\pi f}\right|^2}$$

$$= \frac{35/48}{\left|1 - \frac{7}{12} e^{-j2\pi f} + \frac{1}{6} e^{-j4\pi f}\right|^2}$$

Solution to Problem 6

$\cancel{w(n) =} W_1(n) + w_2(n)$

$W_1(n) = \left\{\frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi}{M}(n+.5)\right)\right\}\left\{u(n) - u(n-M)\right\}$

$\text{DTFT}\left\{u(n) - u(n-M)\right\} = e^{-j\omega\frac{(M-1)}{2}} \dfrac{\sin\left(\frac{M}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$

and:

$\cos(\omega_0 n + \theta)\, x(n) \xleftrightarrow{\text{DTFT}} \frac{e^{j\theta}}{2} X(\omega-\omega_0) + \frac{e^{-j\theta}}{2} X(\omega+\omega_0)$

note: $-\frac{1}{2} \cdot \frac{1}{2} e^{j\frac{2\pi}{M}} \cdot e^{-j\frac{(M-1)}{2}\left(-\frac{4\pi}{M}\right)}$

$\Rightarrow -\frac{1}{4} e^{j\frac{2\pi}{M}} e^{j\pi 2} e^{-j\frac{2\pi}{M}} = -\frac{1}{4}$

So $\text{DTFT}\{W_1(n)\} = e^{-j\frac{(M-1)}{2}\omega}\left\{\frac{1}{2}\frac{\sin\left(\frac{M}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right.$

$\left. - \frac{1}{4}\frac{\sin\left(\frac{M}{2}\left(\omega - \frac{4\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{4\pi}{M}\right)\right)} - \frac{1}{4}\frac{\sin\left(\frac{M}{2}\left(\omega + \frac{4\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega + \frac{4\pi}{M}\right)\right)}\right\}$

i	A_i	L_i	ω_i
1	1/2	M	0
2	-1/4	M	$4\pi/M$
3	-1/4	M	$-4\pi/M$

second part:

$\text{DTFT}\left\{\underbrace{\left[\frac{1}{2} + \frac{1}{2}\cos\left[\frac{4\pi}{M}(n+.5)\right]\right]\left[u(n-\tfrac{M}{4}) - u(n-\tfrac{3M}{4})\right]}_{\triangleq\, w_2(n)}\right\} = ?$

$W_3(n) = w_2(n + \frac{M}{4}) = \left\{\frac{1}{2} - \frac{1}{2}\cos\left[\frac{4\pi}{M}(n+.5)\right]\right\}\left\{u(n) - u(n-\frac{M}{2})\right\}$

since $\cos\left[\frac{4\pi}{M}(n+\frac{M}{4}+.5)\right] = \cos\left[\frac{4\pi}{M}(n+.5) + \pi\right]$

Sol'n. to Prob. 5 (cont.)

$$W_3(\omega) = \text{DTFT}\left\{ \underbrace{\left[\frac{1}{2} - \frac{1}{2}\cos\left[\frac{2\pi}{M/2}(n+.5)\right]\right]\left[u(n) - u(n-\tfrac{M}{2})\right]}_{\text{Simply a Hanning window of length } \frac{M}{2}} \right\} = ?$$

$\Rightarrow$ done in class for length M

Hence:

$$\text{DTFT}\{w_3(n)\} = e^{-j\frac{(\frac{M}{2}-1)}{2}\omega}\left\{\frac{1}{2}\frac{\sin\left(\frac{M}{4}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right.$$

$$\left. + \frac{1}{4}\frac{\sin\left(\frac{M}{4}(\omega - \frac{4\pi}{M})\right)}{\sin\left(\frac{1}{2}(\omega - \frac{4\pi}{M})\right)} + \frac{1}{4}\frac{\sin\left(\frac{M}{4}(\omega + \frac{4\pi}{M})\right)}{\sin\left(\frac{1}{2}(\omega + \frac{4\pi}{M})\right)}\right\}$$

now: $\text{DTFT}\{w_2(n)\} = \text{DTFT}\{w_3(n - \frac{M}{4})\}$

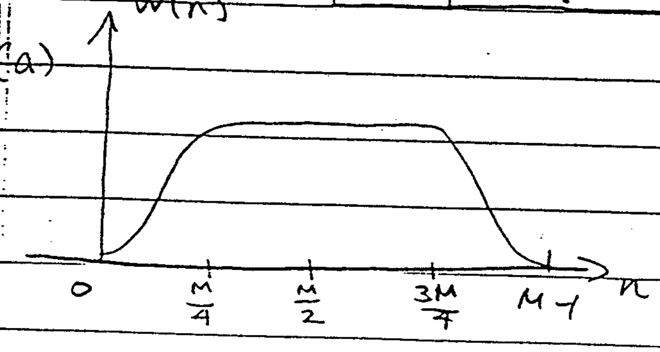
$$= e^{-j\frac{M}{4}\omega}\, W_3(\omega)$$

note: $e^{-j\frac{(\frac{M}{2}-1)}{2}\omega} \cdot e^{-j\frac{M}{4}\omega} = e^{-j\left(\frac{M-1}{2}\right)\omega}$

Thus:

i	A_i	L_i	ω_i
4	1/2	M/2	0
5	1/4	M/2	$4\pi/M$
6	1/4	M/2	$-4\pi/M$

(a)



$$\cos\left[\frac{4\pi}{M}(M-1-n+.5)\right]$$

$$= \cos\left[-\frac{4\pi}{M}(n+.5) + 4\pi\right]$$

$$= \cos\left[\frac{4\pi}{M}(n+.5)\right]$$

Sol'n. to Prob. 6

Notation for parts (c) thru (f)

$w(n) \xleftrightarrow{DTFT} W(\omega)$

$w_R(n) \xleftrightarrow{DTFT} W_R(\omega)$

where $w_R(n)$ is rectangular window of length M

$w_H(n) \xleftrightarrow{DTFT} W_H(\omega)$

where $w_H(n)$ is Hamming window of length M

(c) mainlobe width $W(\omega)$ > mainlobe width $W_R(\omega)$ } $w(n)$ is "narrower" than $w_R(n)$

(d) mainlobe width $W(\omega)$ < mainlobe width $W_H(\omega)$ } $w_H(n)$ more tapered than $w(n)$

(e) peak sidelobe $W(\omega)$ < peak sidelobe $W_R(\omega)$ } $w(n)$ is tapered at edges

(f) peak sidelobe $W(\omega)$ > peak sidelobe $W_H(\omega)$ } $w_H(n)$ is more tapered than $w(n)$