Problem 1. [35 points]

Consider the ARMA(1,1) process generated via the difference equation

$$x[n] = \frac{1}{2}x[n-1] + \nu[n] + \nu[n-1]$$

where $w[n]$ is a stationary white noise process with $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \delta[m]$.

(a) Determine the numerical values of $r_{xx}[0]$, $r_{xx}[1]$, $r_{xx}[2]$, where $r_{xx}[m]$ is the autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$.

(b) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m]$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega}$$

(c) Consider the second-order predictor

$$\hat{x}[n] = -a_2(1)x[n-1] - a_2(2)x[n-2]$$

Determine the numerical values of the optimum predictor coefficients $a_2(1)$ and $a_2(2)$ and the corresponding minimum mean-square error.

Problem 2. [35 points]

Consider the discrete-time complex-valued random process defined for all n:

$$x[n] = D + A_1 e^{j(\omega_1 n + \Theta_1)} + A_2 e^{j(\omega_2 n + \Theta_2)} + \nu[n]$$

where the respective frequencies, ω_1 and ω_2, of the two complex sinewaves are deterministic but unknown constants. The amplitudes, A_1 and A_2, and the constant D are also deterministic but unknown constants. Θ_1 and Θ_2 are independent random variables with each uniformly distributed over a 2π interval and $\nu[n]$ is a stationary random process with zero mean and $r_{\nu\nu}[m] = E\{\nu[n]\nu^*[n-m]\} = \delta[m]$. That is, $\nu[n]$ forms an i.i.d. sequence with a variance of unity. Note, $\nu[n]$ is independent of both Θ_1 and Θ_2 for all n. The values of the autocorrelation sequence for $x[n]$, $r_{xx}[m] = E\{x[n]x^*[n-m]\}$, for three different lag values are given below.

$$r_{xx}[0] = 5, \quad r_{xx}[1] = -1 + j, \quad r_{xx}[2] = 2, \quad r_{xx}[3] = -1 - j$$

Determine the numerical values of ω_1 and ω_2.

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SOLUTIONS

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Problem 1

$$\frac{X(z)}{W(z)} = H(z) = \frac{z+1}{z-\frac{1}{2}}$$

$$r_{xx}[m] = r_{hh}[m] \cdot \underbrace{\sigma_w^2}_{=1} = r_{hh}[m]$$

$$= h[m] * h[-m]$$

$h[n] = ?$

$$z-\tfrac{1}{2}\,\overline{)\,z+1\,}\quad 1$$
$$\underline{-(z-\tfrac{1}{2})}$$
$$\tfrac{3}{2}$$

$$\frac{z+1}{z-\frac{1}{2}} = 1 + \frac{3}{2}\frac{1}{z-\frac{1}{2}} = 1 + \frac{3}{2} z^{-1} \frac{z}{z-\frac{1}{2}}$$

$$h[n] = \delta[n] + \frac{3}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= 3\left(\frac{1}{2}\right)^{n} u[n] - 2\delta[n]$$

$$= \left\{3\left(\frac{1}{2}\right)^{n} u[n] - 2\delta[n]\right\}$$

$$r_{hh}[m] = h[m] * h[-m]$$

$$= \left\{3\left(\frac{1}{2}\right)^{m} u[m] - 2\delta[m]\right\} * \left\{3\left(\frac{1}{2}\right)^{-m} u[-m] - 2\delta[-m]\right\}$$

$$= \left\{9 \underbrace{\frac{1}{1-\left(\frac{1}{2}\right)^2}}_{\frac{4}{3}} \cdot \left(\frac{1}{2}\right)^{|m|} - 6\left(\frac{1}{2}\right)^{m} u[m] - 6\left(\frac{1}{2}\right)^{-m} u[-m] + 4\delta[m]\right\}$$

$\alpha 5$ ②

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Problem 1 (cont.)

$m=0$: $r_{hh}[0] = \{12 - 6 - 6 + 4\} = 4$

$m=1$: $r_{hh}[1] = \{12(\frac{1}{2}) - 6 \cdot \frac{1}{2}\} = 3$

$m=2$: $r_{hh}[2] = \{12(\frac{1}{4}) - 6\frac{1}{4}\} = 3 - \frac{3}{2} = \frac{3}{2}$

(b) $$S_{xx}(\omega) = \underset{\sigma_w^2}{(1)} \frac{|1 + e^{-j\omega}|^2}{|1 - \frac{1}{2}e^{-j\omega}|^2}$$

(c) $$a_1(1) = \frac{-r_{xx}[1]}{r_{xx}[0]} = \frac{-3}{4}$$

$$\mathcal{E}_1 = r_{xx}[0]\{1 - a_1^2(1)\} = 7/4$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = -\begin{bmatrix} 3 \\ 3/2 \end{bmatrix}$$

$$a_1(1) = \frac{-15}{14} \qquad a_2(2) = \frac{3}{7}$$

$$\mathcal{E}_2 = \mathcal{E}_1\left(1 - \left(\frac{3}{7}\right)^2\right) = \frac{10}{7}$$

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Problem 2

$$r_{xx}[m] = -a_1 r_{xx}[m-1] - a_2 r_{xx}[m-2]$$

$$\begin{array}{l} m=1 \\ m=2 \end{array} \quad \begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2-j \\ -2+j & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2-j \\ -1 \end{bmatrix}$$

$$a_1 = 1-j \qquad a_2 = -j$$

$$z^2 + a_1 z + a_2$$

$$= z^2 + \underbrace{(1-j)}_{\text{sum of roots}} z \underbrace{-j}_{\text{product of roots}}$$

$$= (z+1)(z-j)$$

$$-1 = e^{j\pi} \Rightarrow \omega_1 = \pi$$

$$j = e^{j\frac{\pi}{2}} \Rightarrow \omega_2 = \frac{\pi}{2}$$

(b) Sum of p complex sinewaves perfectly predicted from p past values

$$a_1(1) = a_1 = 1-j \qquad a_2(2) = a_2 = -j$$