ECE 538 Exam 3 Solutions Fall '07

1

Sol'n to Prob. 1

$x[n]$ $L=4$ } both causal
$h[n]$ $M=5$ } starting at $n=0$

$y[n] = x[n] * h[n]$ is of length $L+M-1 = 8$

$\Downarrow$

$y[n] \neq 0$ for $n=0,1,\ldots,7$

$y[n]=0$ for $n>7$

6-pt. DFT: $y_6[n] = y[n] + y[n+6]$

$$y_6[0] = y[0] + y[6] = 3$$

$$y_6[1] = y[1] + y[7] = 3$$

$$y_6[2] = y[2] = 3$$

$$y_6[3] = y[3] = 4$$

$$y_6[4] = y[4] = 4$$

$$y_6[5] = y[5] = 3$$

five

5-pt DFT $y_5[n] = y[n] + y[n+5]$

$$y_5[0] = y[0] + y[5] = 4$$

$$y_5[1] = y[1] + y[6] = 4$$

$$y_5[2] = y[2] + y[7] = 4$$

$$y_5[3] = y[3] = 4$$

$$y_5[4] = y[4] = 4$$

Sol'n to Prob. 1 (cont.)

Since $y[5] = 3$:
$$y_5[0] = 4 = y[0] + y[5]$$
$$4 = y[0] + 3$$
$$\Rightarrow y[0] = 1$$

Since $y[0] = 1$:
$$y_6[0] = 3 = y[0] + y[6]$$
$$= 1 + y[6]$$
$$\Rightarrow y[6] = 2$$

Since $y[6] = 2$:
$$y_5[1] = y[1] + y[6]$$
$$4 = y[1] + 2$$
$$\Rightarrow y[1] = 2 \quad \text{(two)}$$

Since $y[1] = 3$:
$$y_6[1] = \underbrace{y[1]}_{\text{(two)}} + y[7]$$
$$3 = 2 + y[7]$$
$$\Rightarrow y[7] = 1$$

THUS:
$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4, 4, 3, 2, \underset{\substack{\uparrow \\ n=7}}{1}\}$$

3

Sol'n to Prob. 2 $\quad x[n] = x[n-1] - \frac{1}{2} x[n-2] + v[n]$

$\sigma_w^2 = \frac{5}{2}$

THUS:

$$r_{xx}[m] = r_{xx}[m-1] - \frac{1}{2} r_{xx}[m-2] + \frac{5}{2} \delta[m]$$

$m=0: \quad r_{xx}[0] = r_{xx}[-1] - \frac{1}{2} r_{xx}[-2] + \frac{5}{2}$

$m=1: \quad r_{xx}[1] = r_{xx}[0] - \frac{1}{2} r_{xx}[-1]$

$m=2: \quad r_{xx}[2] = r_{xx}[1] - \frac{1}{2} r_{xx}[0]$

$\boxed{r_{xx}[-m] = r_{xx}[m]}$

$$\begin{bmatrix} 1 & -1 & 1/2 \\ -1 & (1+\frac{1}{2}) & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix} \begin{bmatrix} r_{xx}[0] \\ r_{xx}[1] \\ r_{xx}[2] \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array}$$

$r_{xx}[0] = -\frac{3}{2} r_{xx}[1] \qquad$ B $\quad -r_{xx}[0] + \frac{3}{2} r_{xx}[1] = 0$

2 A − C :

$$\left(2-\frac{1}{2}\right) r_{xx}[0] + (-2+1)\, r_{xx}[1] = 2 \cdot \frac{5}{2} = 5$$

$$\frac{3}{2} r_{xx}[0] - r_{xx}[1] = 5 \qquad \text{D}$$

B $+ \frac{3}{2}$ D : $\quad \left(-1 + \frac{9}{4}\right) r_{xx}[0] = \frac{15}{2} \Rightarrow \frac{5}{4} r_{xx}[0] = \frac{15}{2}$

$$r_{xx}[0] = \frac{4}{5} \cdot \frac{15}{2} = 3 \cdot 2 = 6$$

$r_{xx}[1] = \frac{2}{3} r_{xx}[0] = 4 \qquad r_{xx}[2] = r_{xx}[1] - \frac{1}{2} r_{xx}[0] = 1$

Sol'n to Prob. 2 (cont.)

Prob. 2

(a) $r_{xx}[0] = 6 \quad r_{xx}[1] = 4 \quad r_{xx}[2] = 1$

(b)
$$S_{xx}(\omega) = \frac{5/2}{\left|1 - e^{-j\omega} + \frac{1}{2}e^{-j2\omega}\right|^2}$$

(c)
$$a_1(1) = -\frac{r_{xx}[1]}{r_{xx}[0]} = \frac{-4}{6} = -\frac{2}{3}$$

$$\begin{aligned}
\mathcal{E}_1^{min} &= \mathcal{E}_0^{min}\left(1 - a_1^2(1)\right) \\
&= r_{xx}[0]\left(1 - \left(-\tfrac{2}{3}\right)^2\right) \\
&= 6\left(\tfrac{5}{9}\right) \\
&= \frac{10}{3} = 3.3333
\end{aligned}$$

(d) Since AR(2) process:

$$a_3(1) = -1, \quad a_3(2) = \frac{1}{2}, \quad a_3(3) = 0$$

and $\mathcal{E}_3^{min} = \mathcal{E}_2^{min} = \sigma_u^2 = \frac{5}{2} = 2.5$

Sol'n to Prob. 3

$$X_8(k) = F_0(k) + W_8^k F_1(k) \qquad k=0,1,2,3$$

$$X_8(k+4) = F_0(k) - W_8^k F_1(k)$$

Clearly: $X(0) = X(4) = X(2) = X(6) = X(3) = X(7)$

$= 0$ all six values are zero

So, for $k=1$: $W_8^1 = \frac{1}{\sqrt{2}}(1-j) = e^{-j\frac{2\pi}{8}}$

$$X(1) = 4 + e^{-j\frac{2\pi}{8}}\left(-4\, e^{+j\frac{\pi}{4}}\right)$$

$$= 4\left(1 - e^{-j\frac{\pi}{4}} e^{+j\frac{\pi}{4}}\right) = 0$$

$$X(1+4) = X(5) = 4(1+1) = 8$$

so $X_8(k) = 8\,\delta(k-5)$, $k = 0, 1, \ldots, 7$

$$x[n] = e^{j\frac{2\pi 5}{8}n}, \quad n = 0, 1, \ldots, 7$$

$$= \cos\left(2\pi \frac{5}{8} n\right) + j \sin\left(2\pi \frac{5}{8} n\right)$$

$$k_1 = k_2 = 5$$