NAME: **2019**
Digital Signal Processing I **Exam 3** **Fall 2019**
25 Nov. 2019

Cover Sheet

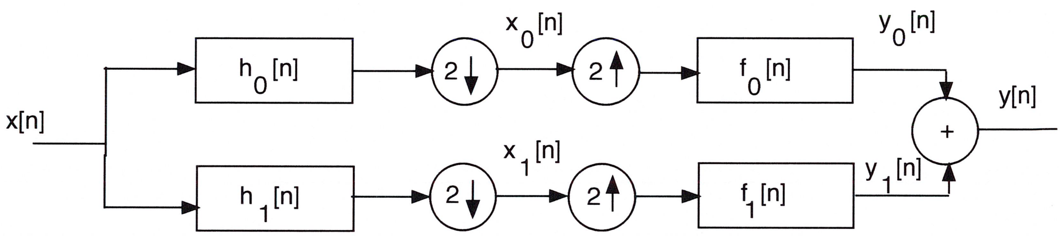
Test Duration: 60 minutes.
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed
Calculators NOT allowed.
This test contains **THREE** problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.

Problem 1. In the system below, the two analysis filters, $h_0[n]$ and $h_1[n]$, and the two synthesis filters, $f_0[n]$ and $f_1[n]$, form a Quadrature Mirror Filter (QMF). Specifically,

$$h_1[n] = (-1)^n h_0[n] \qquad f_0[n] = h_0[n] \qquad f_1[n] = -h_1[n]$$

The lowpass filter $h_0[n]$ employed is the following filter:

$$h_0[n] = 0.5\frac{\sin\left[\frac{\pi}{4}\left(n+\frac{1}{2}\right)\right]}{\pi\left(n+\frac{1}{2}\right)} + 0.5\frac{\sin\left[\frac{3\pi}{4}\left(n+\frac{1}{2}\right)\right]}{\pi\left(n+\frac{1}{2}\right)}$$



Determine mathematically and graphically (include as much detail as possible) if the lowpass half-band filter above satisfies the condition required for Perfect Reconstruction. Be sure to clearly state what that condition is (don't need to rederive it) and then show whether it is satisfied with the filter $h_0[n]$, showing as much detail as possible. **VIP:** for the sake of time, you can (correctly) assume that the net effect of the shift to the left by 0.5 in (discrete) time causes multiplication by the term $e^{j0.5\omega}$ in the frequency domain.

condition to check:

$$H_0^2(\omega) - H_0^2(\omega - \pi)$$

$$H_0(\omega) = e^{j\frac{\omega}{2}}\left(\text{rect}\left(\frac{\omega}{\pi/2}\right) + .5\ \text{rect}\left(\frac{|\omega| - \pi/2}{\pi/2}\right)\right) \Bigg\}$$ periodic w period 2π

for $-\pi < \omega < \pi$

$$H_0(\omega) = e^{j\frac{\omega}{2}} H_r(\omega)$$

$$H_0^2(\omega) - H_0^2(\omega-\pi) = e^{j\frac{\omega}{2}2} H_r^2(\omega) - e^{j\left(\frac{\omega-\pi}{2}\right)2} H_r^2(\omega-\pi)$$

$$= e^{j\omega}\left\{H_r^2(\omega) + H_r^2(\omega-\pi)\right\}$$

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For perfect reconstruction: $\frac{1}{2}\left\{ H_0(\omega)^2 - H_0(\omega-\pi)^2 \right\} = ce^{-jn_0\omega}$

$$H_0(\omega) = \frac{1}{2} e^{j\frac{1}{2}\omega} \operatorname{rect}\left(\frac{\omega}{2\cdot\frac{\pi}{4}}\right) + \frac{1}{2} e^{j\frac{1}{2}\omega} \operatorname{rect}\left(\frac{\omega}{2\cdot\frac{3\pi}{4}}\right) = \frac{1}{2} e^{j\frac{1}{2}\omega}\left[\operatorname{rect}\left(\frac{\omega}{2\cdot\frac{\pi}{4}}\right) + \operatorname{rect}\left(\frac{\omega}{2\cdot\frac{3\pi}{4}}\right)\right]$$

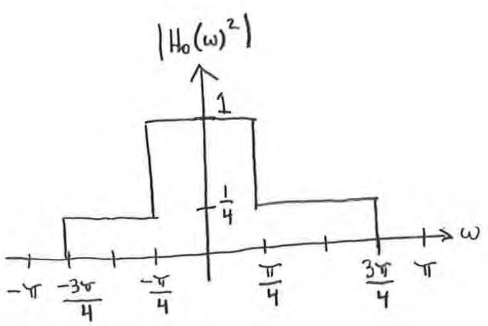
$$H_0(\omega) = \begin{cases} 0 & \text{for } -\pi < \omega < \frac{-3\pi}{4} \\ \frac{1}{2} e^{j\frac{1}{2}\omega} & \text{for } \frac{-3\pi}{4} < \omega < \frac{-\pi}{4} \\ e^{j\frac{1}{2}\omega} & \text{for } \frac{-\pi}{4} < \omega < \frac{\pi}{4} \\ \frac{1}{2} e^{j\frac{1}{2}\omega} & \text{for } \frac{\pi}{4} < \omega < \frac{3\pi}{4} \\ 0 & \text{for } \frac{3\pi}{4} < \omega < \pi \end{cases}$$

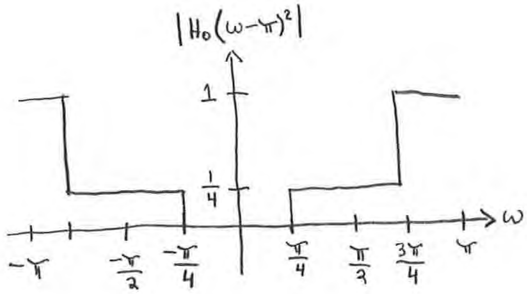
for $0 < \omega < \frac{\pi}{4}$: $\frac{1}{2}\left\{ \left(e^{j\frac{1}{2}\omega}\right)^2 - (0)^2 \right\} = \frac{1}{2} e^{j\omega}$

for $\frac{\pi}{4} < \omega < \frac{3\pi}{4}$: $\frac{1}{2}\left\{ \left(\frac{1}{2} e^{j\frac{1}{2}\omega}\right)^2 - \left(\frac{1}{2} e^{j\frac{1}{2}(\omega-\pi)}\right)^2 \right\}$

$$= \frac{1}{2}\left\{ \frac{1}{4} e^{j\omega} - \frac{1}{4} e^{j(\omega-\pi)} \right\} = \frac{1}{2}\left\{ \frac{1}{2} e^{j\omega} \right\} = \frac{1}{4} e^{j\omega}$$

for $\frac{3\pi}{4} < \omega < \pi$: $\frac{1}{2}\left\{ (0)^2 - \left(e^{j\frac{1}{2}(\omega-\pi)}\right)^2 \right\} = \frac{-1}{2} e^{j(\omega-\pi)} = \frac{1}{2} e^{j\omega}$





This filter does not satisfy the condition for perfect reconstruction because $\frac{1}{2}\left\{ H_0(\omega)^2 - H_0(\omega-\pi)^2 \right\}$ does not have constant magnitude

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for $0 < \omega < \frac{\pi}{4}$: $H_r^2(\omega) + H_r^2(\omega - \pi) = H_r^2(\omega)$

$= 1^2 = 1 \Rightarrow$ OK

$\frac{3\pi}{4} < \omega < \pi$: $H_r^2(\omega) + H_r^2(\omega - \pi) = H_r^2(\omega - \pi)$

$= 1^2 \Rightarrow$ OK

BUT:

$\frac{\pi}{4} < \omega < \frac{3\pi}{4}$: $H_r^2(\omega) + H_r^2(\omega - \pi)$

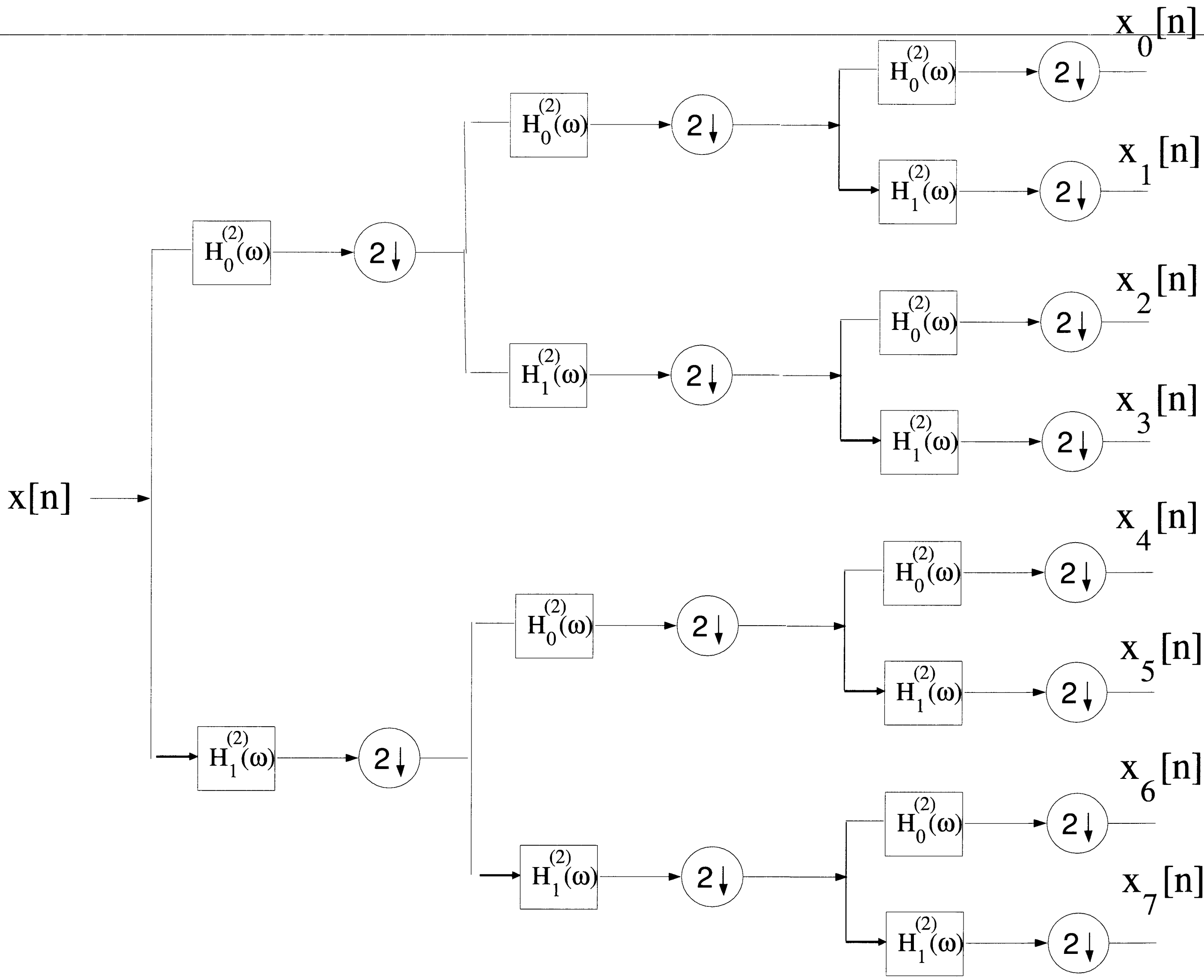
$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

DOES NOT WORK $\Rightarrow$ NO Perfect Reconstruction

If the relative weightings are changed to:

$$h_0[n] = \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\sin\left(n\frac{\pi}{4}\right)}{n\pi} + \frac{1}{\sqrt{2}} \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n}$$

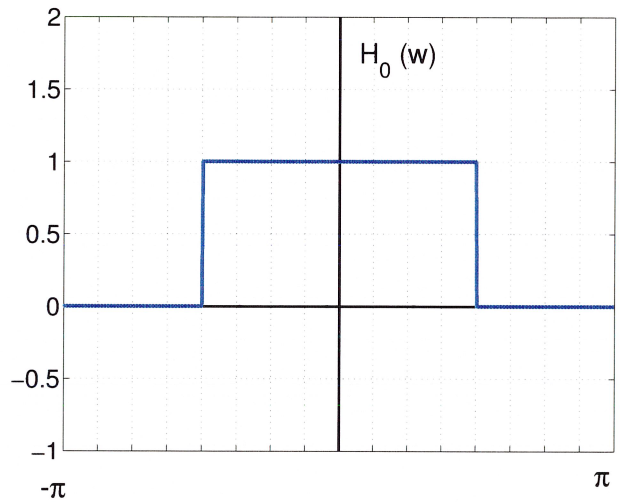
$\Rightarrow$ this works!


x[n]
H_0^(2)(ω)
H_1^(2)(ω)
2↓
x_0[n]
x_1[n]
x_2[n]
x_3[n]
x_4[n]
x_5[n]
x_6[n]
x_7[n]

Problem 2. In Matlab Homework 2, you used Noble's Identities to convert the Tree-Structured subbander on the previous page to the regular maximally decimated subbander, obtaining products of the form:

$$H_k(\omega) = H_\ell^{(2)}(\omega)H_m^{(2)}(2\omega)H_n^{(2)}(4\omega) \quad k = 0, 1, ..., 7$$

where $\ell \in \{0,1\}$, $m \in \{0,1\}$, and $n \in \{0,1\}$, with $H_0^{(2)}(\omega)$ denoting the lowpass halfband filter below for the two-channel QMF and $H_1^{(2)}(\omega) = H_0^{(2)}(\omega - \pi)$. For this problem, $H_0^{(2)}(\omega)$ is the ideal lowpass filter below (note: a DTFT is always periodic with period 2π.)



You are required to fill in the table below. **FIRST** you should plot $H_0^{(2)}(2\omega)$ and $H_0^{(2)}(4\omega)$ on the next page, and then plot $H_1^{(2)}(\omega)$, $H_1^{(2)}(2\omega)$ and $H_1^{(2)}(4\omega)$ on the next page after that, in the space provided. In each plot, the abscissa range is from $-\pi$ to $+\pi$ and there is a tic mark and vertical dashed line at every integer multiple of $\pi/8$. These plots will help you fill in the table. Only fill in the positive frequency band that is passed; the filters are real-valued and even-symmetric, so their respective frequency responses are real-valued and even-symmetric.

$H_1^{(2)}(\omega)H_0^{(2)}(2\omega)H_0^{(2)}(4\omega)$ passes: $7\pi/8 \le \omega < \pi$

$H_1^{(2)}(\omega)H_0^{(2)}(2\omega)H_1^{(2)}(4\omega)$ passes: $6\pi/8 < \omega < 7\pi/8$

$H_1^{(2)}(\omega)H_1^{(2)}(2\omega)H_1^{(2)}(4\omega)$ passes: $5\pi/8 < \omega < 6\pi/8$

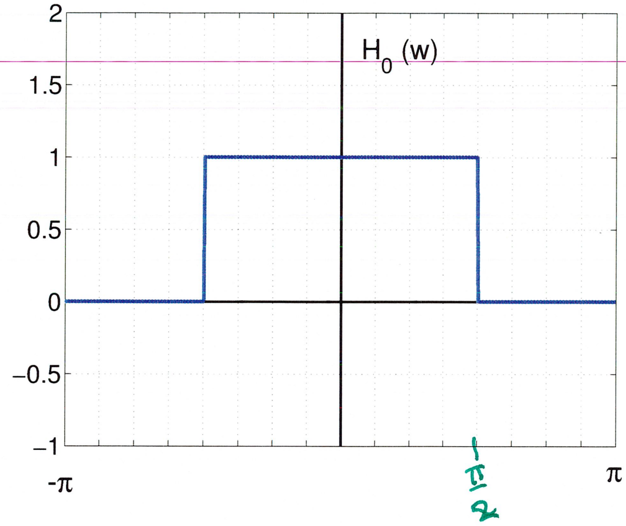
$H_1^{(2)}(\omega)H_1^{(2)}(2\omega)H_0^{(2)}(4\omega)$ passes: $4\pi/8 < \omega < 5\pi/8$

$H_0^{(2)}(\omega)H_1^{(2)}(2\omega)H_0^{(2)}(4\omega)$ passes: $3\pi/8 < \omega < 4\pi/8$ (1)

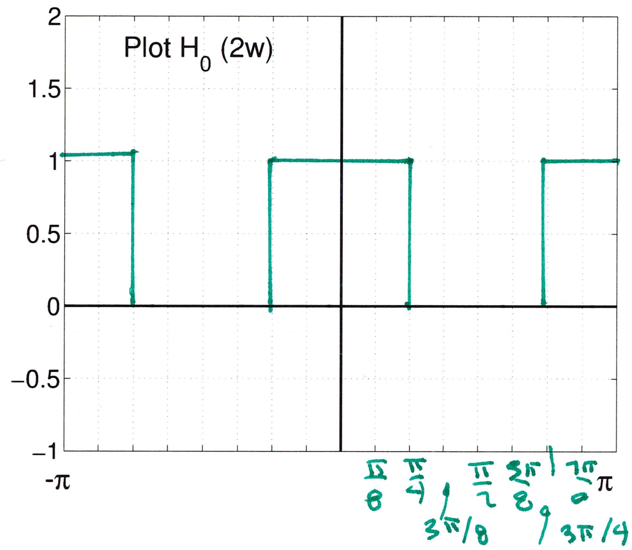
$H_0^{(2)}(\omega)H_1^{(2)}(2\omega)H_1^{(2)}(4\omega)$ passes: $2\pi/8 < \omega < 3\pi/8$

$H_0^{(2)}(\omega)H_0^{(2)}(2\omega)H_1^{(2)}(4\omega)$ passes: $\pi/8 < \omega < 2\pi/8$

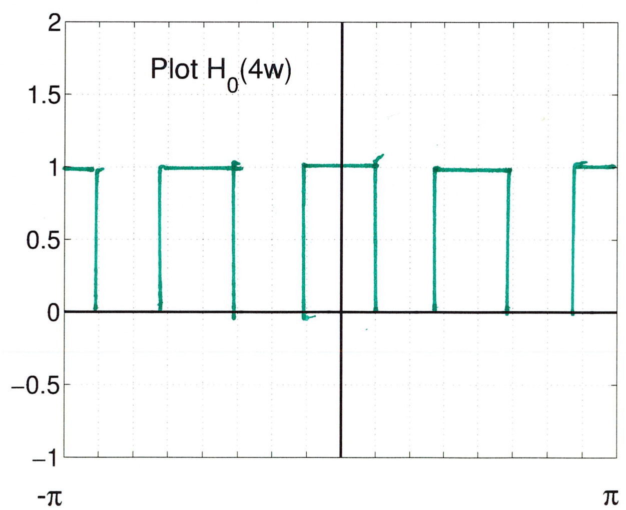
$H_0^{(2)}(\omega)H_0^{(2)}(2\omega)H_0^{(2)}(4\omega)$ passes: $0 < \omega < \pi/8$



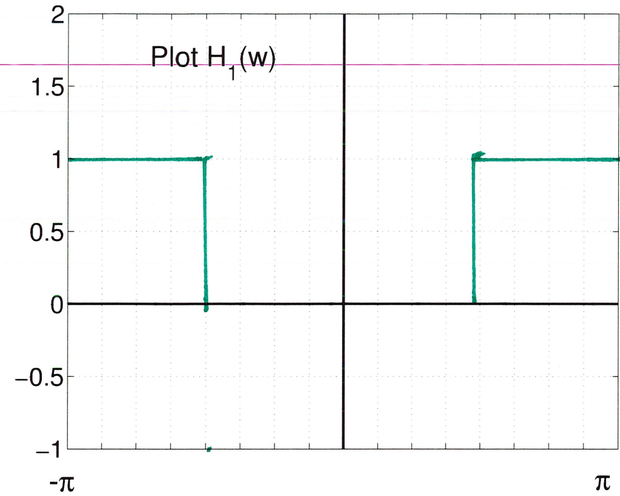
H_0 (w)
-π
π
$\frac{\pi}{2}$



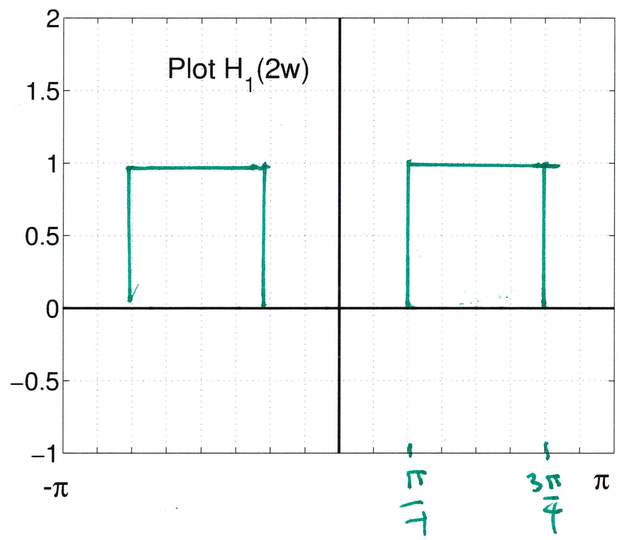
Plot H_0 (2w)
-π
π



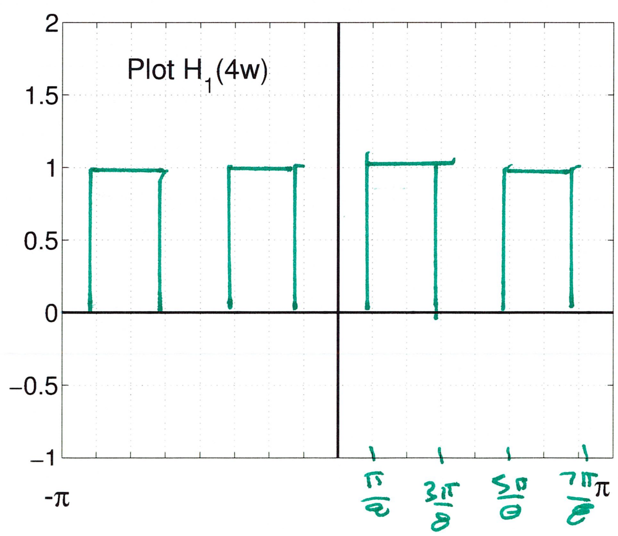
Plot H_0(4w)
-π
π



Plot $H_1(w)$
2
1.5
1
0.5
0
−0.5
−1
-π
π



Plot $H_1(2w)$
2
1.5
1
0.5
0
−0.5
−1
-π
$\frac{\pi}{4}$
$\frac{3\pi}{4}$
π



Plot $H_1(4w)$
2
1.5
1
0.5
0
−0.5
−1
-π
$\frac{\pi}{8}$
$\frac{3\pi}{8}$
$\frac{5\pi}{8}$
$\frac{7\pi}{8}$
π

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Problem 3. Consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin\left[\pi\left(n + \frac{1}{2} + \ell 8\right)\right]}{\pi\left(n + \frac{1}{2} + \ell 8\right)} \{u[n] - u[n-8]\}$$

(a) Determine the 8-pt DFT of $h[n]$, denoted $H_8(k)$, for $0 \le k \le 7$. EITHER write an expression for $H_8(k)$ OR list the numerical values: $H_8(0) =?$, $H_8(1) =?$, $H_8(2) =?$, $H_8(3) =?$, $H_8(4) =?$, $H_8(5) =?$, $H_8(6) =?$, $H_8(7) =?$.

(b) Consider the finite-length sinewave $x[n]$ below is input to the length-8 filter $h[n]$ above, yielding $y[n] = x[n] * h[n]$. Write a closed-form expression for the time-domain aliased output $y_8[n] = y[n] + y[n+8]$, for $n = 0, 1, ..., 7$.

$$x[n] = \left(\cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right)\right)\{u[n] - u[n-8]\}$$

(c) Consider that the infinite-length analog signal below is sampled at a rate of $F_s = 16KHz = 16000$ samples/sec to form the DT signal $x[n] = x_a(n/F_s)$. The infinite-length DT signal is then input to the length-8 filter $h[n]$ defined in part (a). Determine and write an expression for the output $y_1[n] = x[n] * h[n]$.

$$x_a(t) = \cos(2\pi 2000t) + \cos(2\pi 4000t) + \cos(2\pi 6000t)$$

(d) Your answer to part (c), $y_1[n]$ is interleaved with $x[n]$ to form $y[n]$, as indicated below. Write a closed-form expression for $y[n]$.

$$\{..., x[-2], y_1[-2], x[-1], y_1[-1], x[0], y_1[0], x[1], y_1[1], x[2], y_1[2], ...\}$$

(a) $H_8(k) = e^{j\frac{\omega}{2}}\Big|_{\omega = k2\pi/8}$ for $0 \le k < 4$

$-e^{j\frac{\omega}{2}} = e^{j\frac{(\omega - 2\pi)}{2}}\Big|_{\omega = k2\pi/8}$ for $4 < k \le 7$

$k=4$ $H_8(k) = 0 \Rightarrow$ discontinuity at $\omega = \pi$

accept $e^{j\frac{\pi}{2}}$ or $e^{-j\frac{\pi}{2}}$ as well

$$e^{j\frac{1}{2}k\frac{2\pi}{8}} = e^{jk\frac{\pi}{8}}$$

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$$H_8(k) = \left\{1, e^{j\frac{\pi}{8}}, e^{j\frac{2\pi}{8}}, e^{j\frac{3\pi}{8}}, 0, -e^{j\frac{5\pi}{8}}, -e^{j\frac{6\pi}{8}}, -e^{j\frac{7\pi}{8}}\right\}$$

$k=0 \quad k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5 \quad k=6 \quad k=7$

(b) two lucky cases (4 lucky complex-valued cases)

$$y_8[n] = \cos\left(\underbrace{\frac{2\pi}{8}}_{k=1} n + \angle H_8(1)\right) + \cos\left(\underbrace{2}_{k=2}\frac{2\pi}{8} n + \angle H_8(2)\right)$$

$$= \cos\left(\frac{\pi}{4}n + \frac{\pi}{8}\right) + \cos\left(\frac{\pi}{2}n + \frac{2\pi}{8}\right) \quad n = 0, 1, \ldots, 7$$

(c) $x[n] = \cos\left(\frac{2\pi}{8}n\right) + \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{3\pi}{4}n\right)$

$-\infty < n < \infty$

$$= \cos\left(\frac{\pi}{4}n + \angle H_8(1)\right) + \cos\left(\frac{\pi}{2}n + \angle H_8(2)\right) + \cos\left(\frac{3\pi}{4}n + \angle H_8(3)\right)$$

$$= \cos\left(\frac{\pi}{4}n + \frac{\pi}{8}\right) + \cos\left(\frac{\pi}{2}n + \frac{2\pi}{8}\right) + \cos\left(\frac{3\pi}{4}n + \frac{3\pi}{8}\right)$$

(d) Upsampling by 2x:

$$y[n] = \cos\left(\frac{\pi}{8}n\right) + \cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{3\pi}{8}n\right)$$

each frequency is halved $\Big\}$ due to doubling the sampling rate