SOLUTION

**NAME:**
**Digital Signal Processing I** **Exam 3** **2018**
**Session 40** **Fall 2018**
**30 Nov. 2018**

## Cover Sheet

Test Duration: 60 minutes.
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed
Calculators NOT allowed.
Show all work. More credit for approach than final answer.
This test contains **THREE** problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.

**GIVEN NOBLE's IDENTITIES TO USE IN PROBLEM 1.**

(a) If $E(\omega)$ in Figure 1(b) in terms of $G(\omega)$ in Figure 1(a) satsifies $E(\omega) = G(M\omega)$, the I/O relationship of the system in Figure 1(b) is exactly the same as the I/O relationship of the system in Figure 1(a). This result is known as Noble's First Identity.

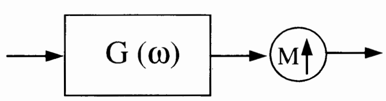


Figure 1(a).

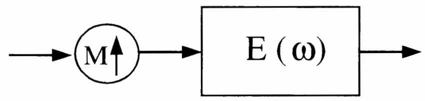


Figure 1(b).

(b) If $F(\omega)$ in Figure 2(b) in terms of $H(\omega)$ in Figure 2(a) satisfies $F(\omega) = H(M\omega)$, the I/O relationship of the system in Figure 2(b) is exactly the same as the I/O relationship of the system in Figure 2(a). This result is known as Noble's Second Identity.

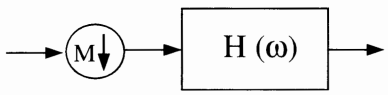


Figure 2(a).

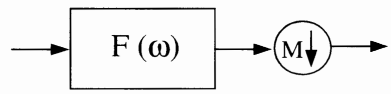


Figure 2(b).

**Problem 1.**

(a) Determine the impulse response $h[n]$ in Figure 3(b) so that the I/O relationship of the system in Figure 3(b) is exactly the same as the I/O relationship of the system in Figure 3(a). Plot the magnitude AND the phase (two separate plots) of the DTFT of $h[n]$ over $-\pi < \omega < \pi$. *Hint:* Analyze the system of Figure 3(a) in the frequency domain using Noble's First Identity.

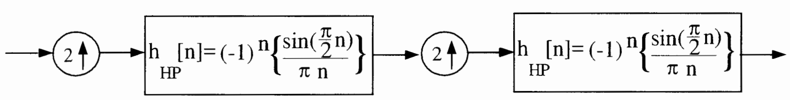


Figure 3(a).

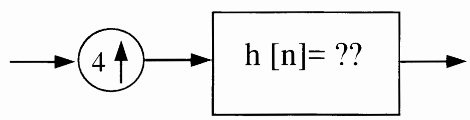


Figure 3(b).

(b) Determine the numerical values of the impulse response $h_{\text{eq}}[n]$ in Figure 4(b) so that the I/O relationship of the system in Figure 4(b) is exactly the same as the I/O relationship of the system in Figure 4(a). *Hint:* Analyze the system of Figure 4(a) in the time domain using Noble's First Identity.

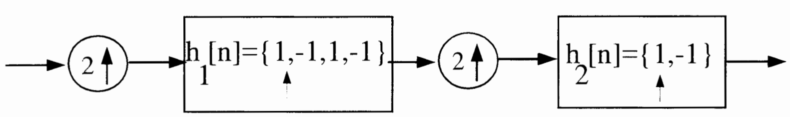


Figure 4(a).

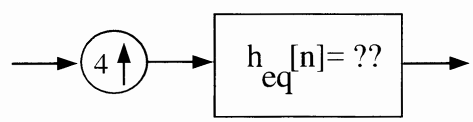


Figure 4(b).

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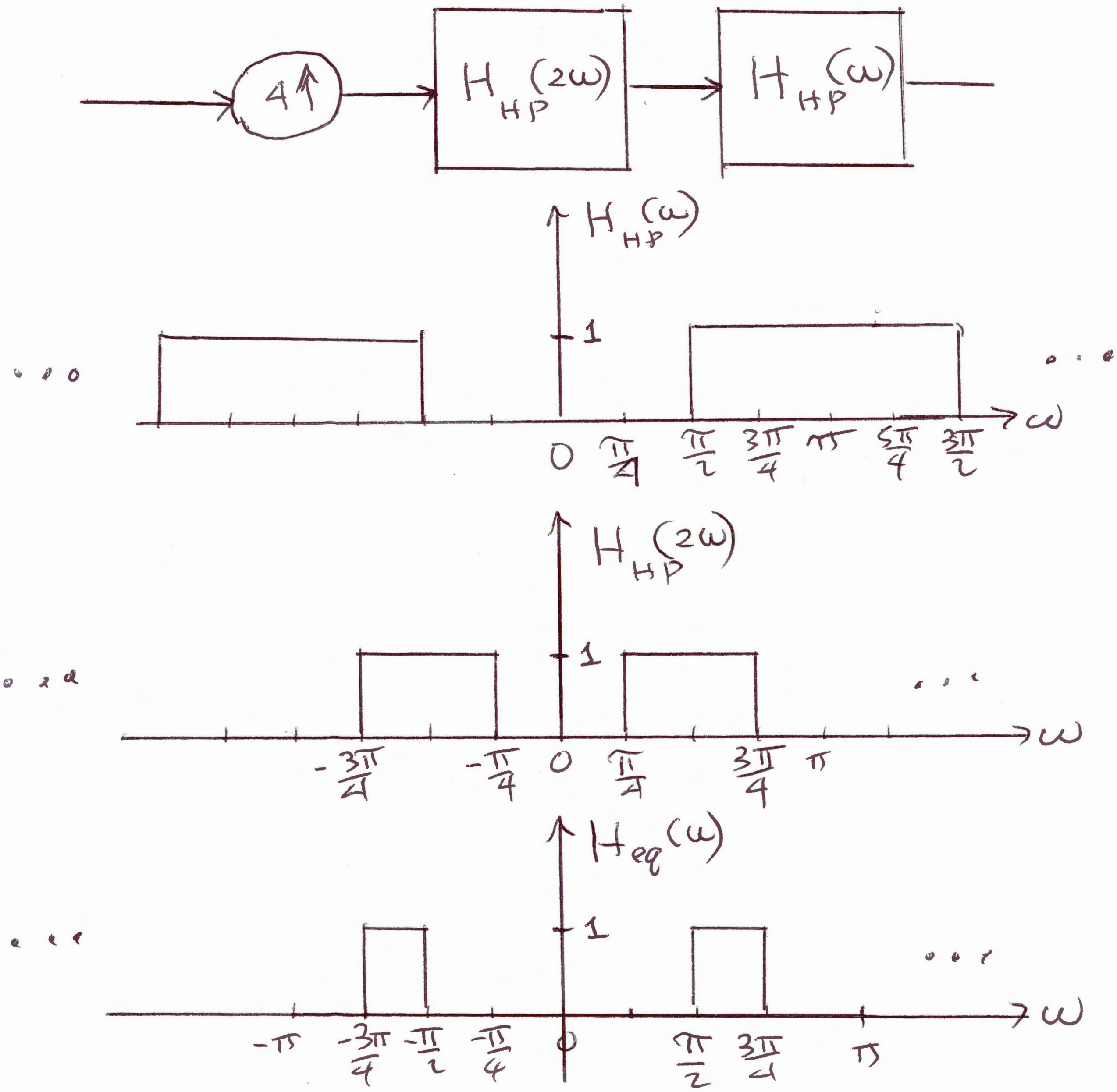


$$h_{eq}[n] = h[n] = 2\,\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\cos\left(\frac{5\pi}{8}n\right)$$

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1 (b) Same procedure as in part (a)

$$H_{eq}(\omega) = H_1(2\omega)\, H_2(\omega)$$

$$h_{eq}[n] = \left\{ \sum_{k=0}^{3} h_1[k]\, \delta[n-2k] \right\} * h_2[n]$$

$$h_{eq}[n] = \{1, 0, -1, 0, 1, 0, -1\} * \{1, -1\}$$

$$\begin{array}{rcccccccc} = & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \\ & & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ \hline \end{array}$$

$$h_{eq}[n] = \{\underset{\uparrow}{1}, -1, -1, 1, 1, -1, -1, 1\}$$

**Problem 2.** Consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h[n] = 2\sin\left(\frac{3\pi}{4}n\right)\{u[n] - u[n-8]\}$$

Consider a DT sinewave $x[n]$ of length $N = 16$ as defined below:

$$x[n] = \left\{\cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{12\pi}{16}n\right)\right\}\{u[n] - u[n-16]\}$$

$y[n]$ is formed as the linear convolution of $x[n]$ with $h[n]$ as:

$$y[n] = x[n] * h[n]$$

We then take the last $M - 1 = 7$ values of $y[n]$ and time-domain alias them into the first $\boxed{\text{seven}}$ values to form a sequence of length 16, denoted $y_a[n]$, according to:

$$y_a[n] = y[n] + y[n+16], n = 0, 1, 2, ..., 6 \quad (1)$$
$$y_a[n] = y[n], n = 7, 8, 9, 10, 11, 12, 13, 14, 15 \quad (2)$$

Determine an expression for $y_a[n]$ similar to the expression for $x[n]$ above. Show all work. You do NOT have to list the 16 numerical values of $y_a[n], n = 0, 1, ..., 15$ in sequence form. **NOTE 1:** Using concepts learned in class; There won't be a lot of points for brute force calculations that do not show what you learned in class for Exam 3.

$$y_a[n] = \left|H\left(\tfrac{\pi}{2}\right)\right| \cos\left(\tfrac{\pi}{2}n + \angle H\left(\tfrac{\pi}{2}\right)\right)\{u[n] - u[n-16]\}$$
$$+ \left|H\left(\tfrac{3\pi}{4}\right)\right| \cos\left(\tfrac{3\pi}{4}n + \angle H\left(\tfrac{3\pi}{4}\right)\right)\{u[n] - u[n-16]\}$$

$$\cos\left(\tfrac{\pi}{2}n\right) = \cos\left((4)\tfrac{2\pi}{16}n\right) \qquad k_0 = 4$$
$$\cos\left(\tfrac{12\pi}{16}n\right) = \cos\left((6)\tfrac{2\pi}{16}n\right) \qquad k_0 = 6$$
$$= \cos\left(\tfrac{3\pi}{4}n\right)$$

"lucky" cases

$$h[n] \xleftrightarrow{DTFT} H(\omega) = \frac{1}{j}\frac{\sin\left(\frac{8}{2}\left(\omega - \frac{3\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{3\pi}{4}\right)\right)} e^{-j\frac{(8-1)}{2}\left(\omega - \frac{3\pi}{4}\right)}$$
$$- \frac{1}{j}\frac{\sin\left(\frac{8}{2}\left(\omega + \frac{3\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega + \frac{3\pi}{4}\right)\right)} e^{-j\frac{(8-1)}{2}\left(\omega + \frac{3\pi}{4}\right)}$$

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$$H\left(\frac{3}{4}\pi\right) = -8j$$

$$H\left(\frac{\pi}{2}\right) = 0$$

$$y_a[n] = 8\cos\left(\underbrace{\frac{3\pi}{4}}_{\frac{12\pi}{16}} n - \frac{\pi}{2}\right)\left\{u[n] - u[n-16]\right\}$$

**Problem 3.** Consider a causal FIR filter of length $M = 9$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \left\{ 2\frac{\sin\left[\frac{\pi}{4}(n+\ell 9)\right]}{\pi(n+\ell 9)} - 4\frac{\sin\left[\frac{\pi}{2}(n+\ell 9)\right]}{\pi(n+\ell 9)} + 3\frac{\sin\left[\frac{3\pi}{4}(n+\ell 9)\right]}{\pi(n+\ell 9)} \right\} \{u[n] - u[n-9]\}$$

(a) Determine the 9-pt DFT of $h_p[n]$, denoted $H_9(k)$, for $0 \le k \le 9$. Write your answer in sequence form to indicate the numerical values of $H_9(k), k = 0, 1, ..., 8$.

(b) Consider the sequence $x[n]$ of length $L = 9$ below.

$\frac{4\pi}{3} = \frac{12\pi}{9}$

$$x[n] = \left\{ -\cos\left(\frac{4\pi}{9}n\right) + 2\sin\left(\frac{8\pi}{9}n\right) + \frac{1}{3}\cos\left(\frac{4\pi}{3}n\right) \right\} \{u[n] - u[n-9]\}$$

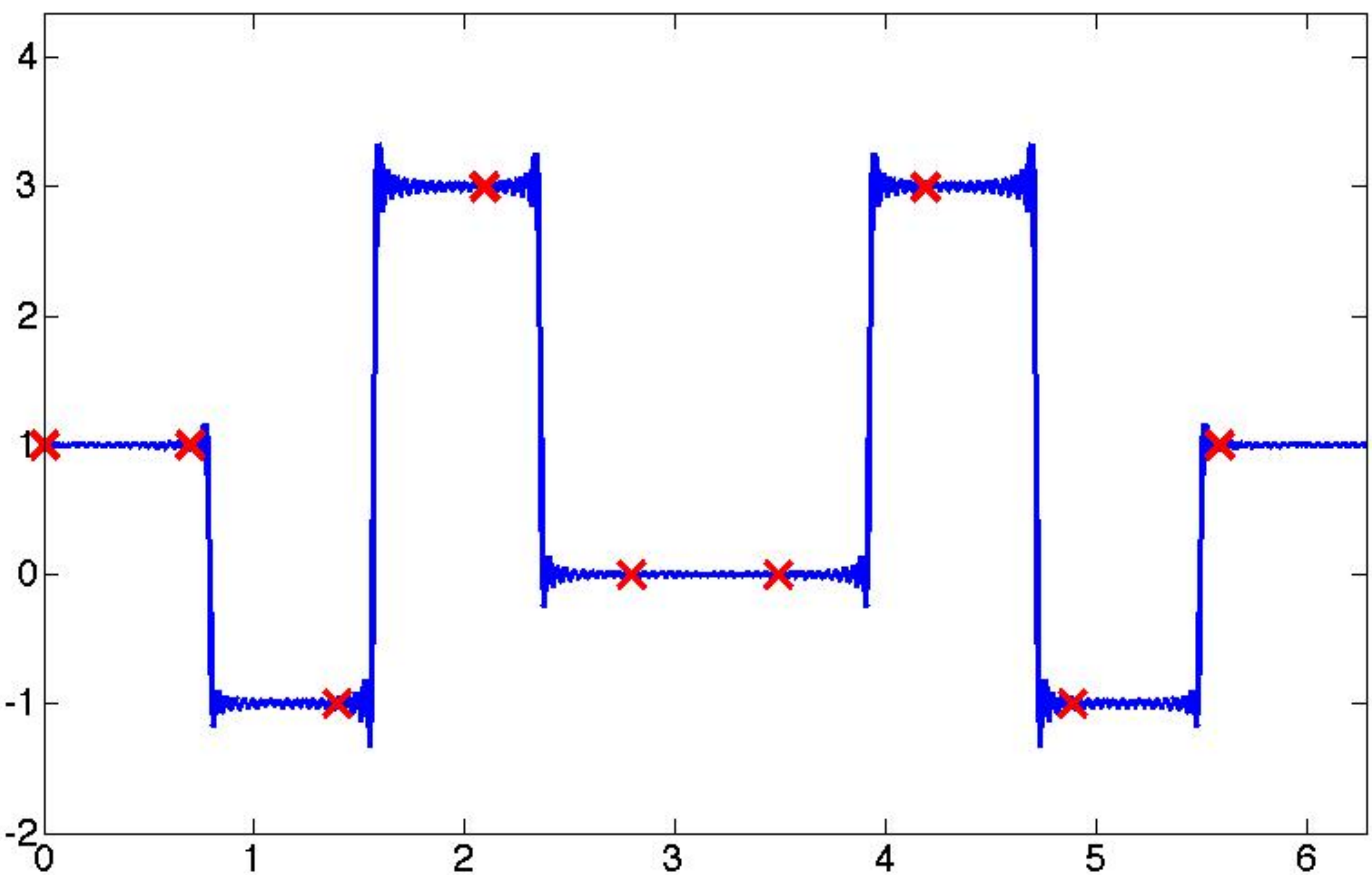
$y_9[n]$ is formed by computing $X_9(k)$ as a 9-pt DFT of $x[n]$, $H_9(k)$ as a 9-pt DFT of $h[n]$ and, finally, then $y_9[n]$ is computed as the 9-pt inverse DFT of $Y_9(k) = X_9(k)H_9(k)$. Express the result $y_9[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

$$h_p[n] \overset{DFT}{\underset{9}{\longleftrightarrow}} \{1, 1, -1, 3, 0, 0, 3, -1, 1\}$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\omega$ | 0 | $\frac{2\pi}{9}$ | $\frac{4\pi}{9}$ | $\frac{6\pi}{9}$ | $\frac{8\pi}{9}$ | $\frac{10\pi}{9}$ | $\frac{12\pi}{9}$ | $\frac{14\pi}{9}$ | $\frac{16\pi}{9}$ |
| $H_p(k)$ | 1 | 1 | -1 | 3 | 0 | 0 | 3 | -1 | 1 |

$$y_9[n] = \underbrace{+}_{(-1)(-1)} \cos\left(\frac{4\pi}{9}n\right) + 0 \cdot 2\sin\left(\frac{8\pi}{9}n\right) + 3\left(\frac{1}{3}\right)\cos\left(\frac{12\pi}{9}n\right)$$

$$= \left\{ \cos\left(\frac{4\pi}{9}n\right) + \cos\left(\frac{4\pi}{3}n\right) \right\} \{u[n] - u[n-9]\}$$


4
3
2
1
0
-1
-2
0
1
2
3
4
5
6