NAME: SOLUTION **2017**
Digital Signal Processing I **Exam 3** **Fall 2017**
Session 40 **1 Dec. 2017**

Cover Sheet

Test Duration: 60 minutes.
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed
Calculators NOT allowed.
Show all work. More credit for approach than final answer.
This test contains **THREE** problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.

Problem 1. Determine the impulse response $h_a[n]$ in Figure 1(b) so that the I/O relationship of the system in Figure 1(b) is exactly the same as the I/O relationship of the system in Figure 1(a). You should obviously work in the frequency domain and show all work. Plot the magnitude of the DTFT of $h_a[n]$ over $-\pi < \omega < \pi$.

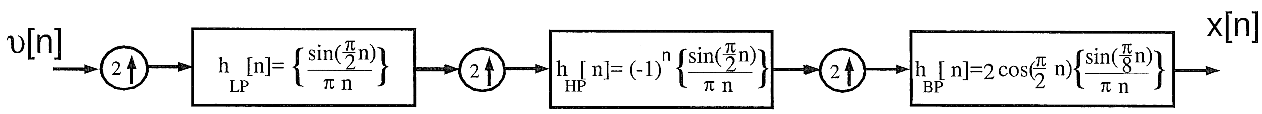


Figure 1(a).

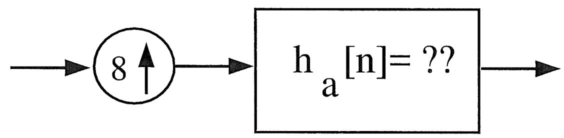
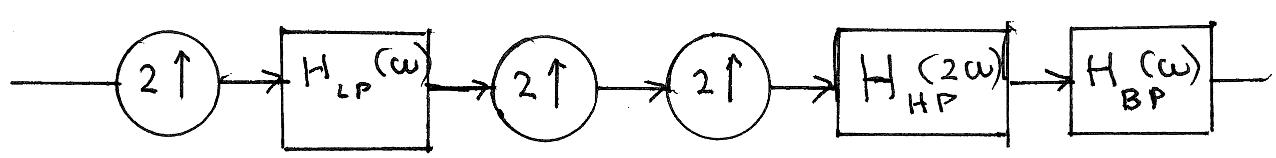
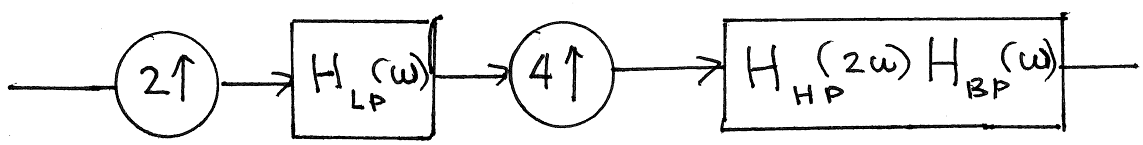
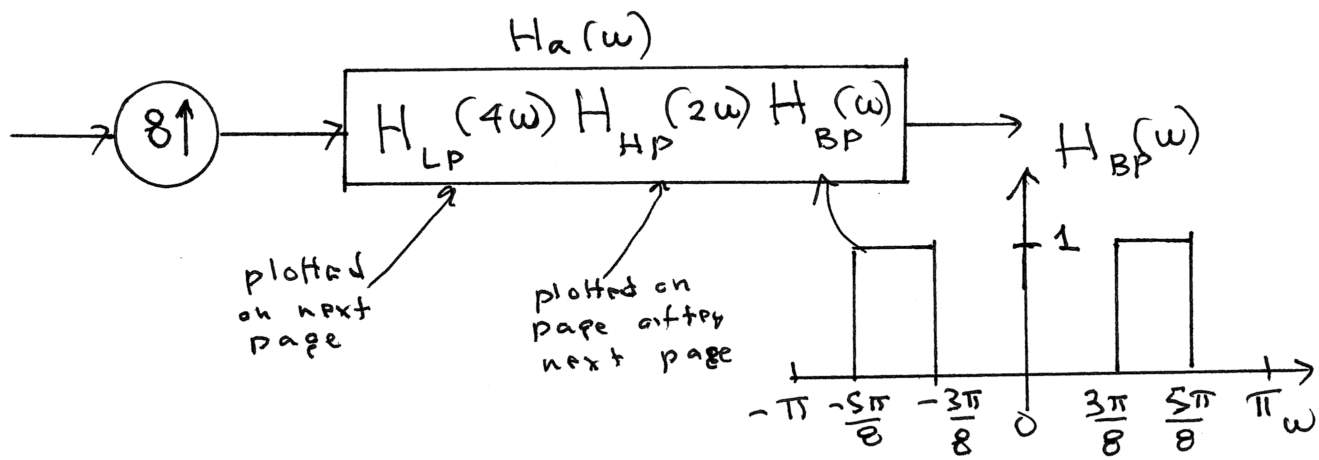


Figure 1(b).

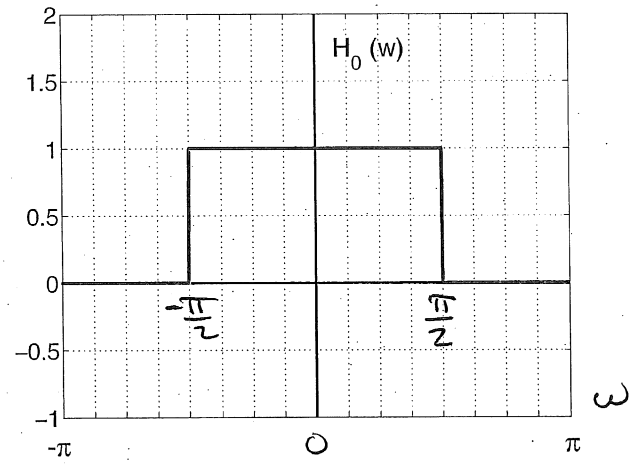


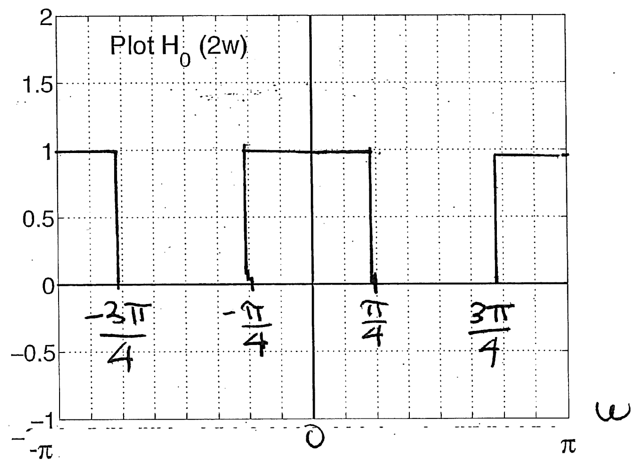


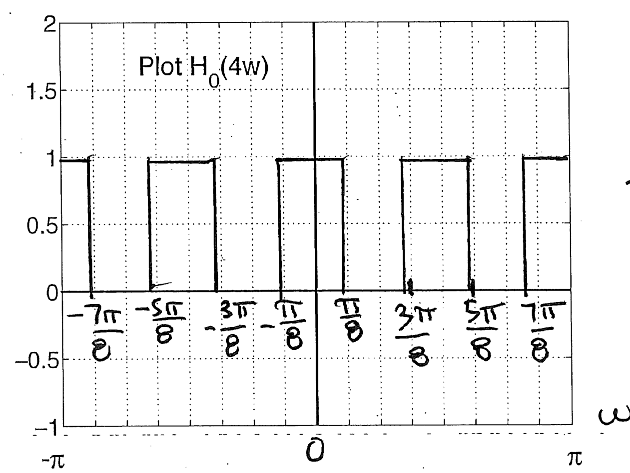


$$H_a(\omega) = H_{LP}(4\omega) H_{HP}(2\omega) H_{BP}(\omega) = H_{BP}(\omega)$$

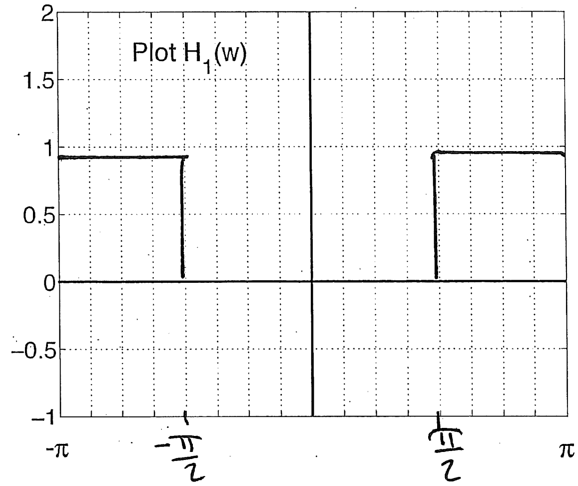
$$h_a(n) = h_{BP}(n) = 2\cos\left(\frac{\pi}{2}n\right)\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}$$

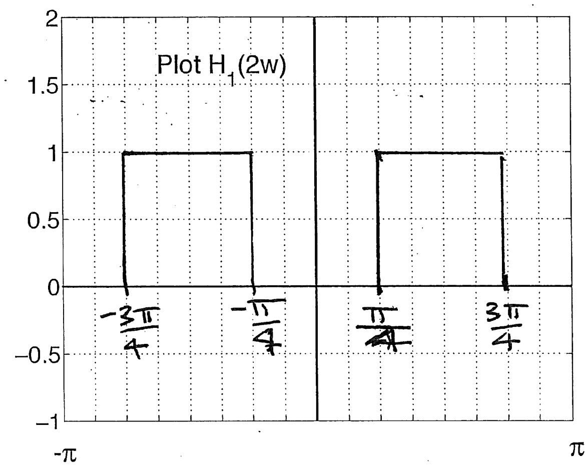




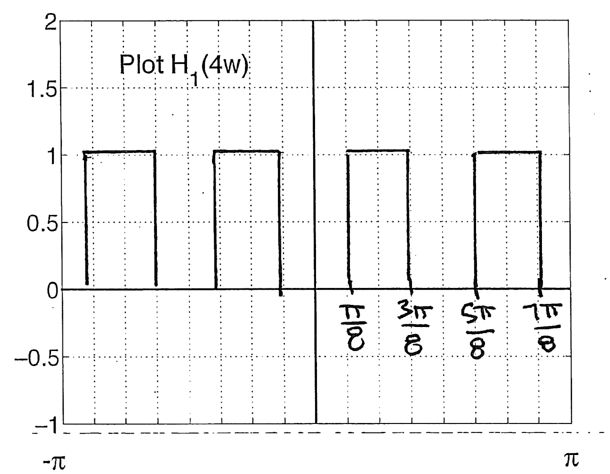


This is $H_{LP}(4w)$





This is $H_{Hp}(2\omega)$



Problem 2. Consider a causal FIR filter of length $M = 4$ with impulse response as defined below:

$$h[n] = \cos\left(\frac{\pi}{4}n\right)\{u[n] - u[n-4]\}$$

Consider a DT sinewave $x[n]$ of length $N = 8$ as defined below:

$$x[n] = \sin\left(\frac{2\pi}{8}n\right)\{u[n] - u[n-8]\}$$

$y[n]$ is formed as the linear convolution of $x[n]$ with $h[n]$ as:

$$y[n] = x[n] * h[n]$$

We then take the last $M - 1 = 3$ values of $y[n]$ and time-domain alias them into the first three values to form a sequence of length 8, denoted $y_a[n]$, according to:

$$y_a[n] = y[n] + y[n+8], n = 0, 1, 2 \quad (1)$$
$$y_a[n] = y[n], n = 3, 4, 5, 6, 7 \quad (2)$$

Determine an expression for $y_a[n]$ similar to the expression for $x[n]$ above; list the 8 numerical values of $y_a[n], n = 0, 1, ..., 7$ in sequence form.
NOTE 1: the trigonometric values below are useful in this problem AND in Problem 3 as well:

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \underset{\sin}{\cancel{\cos}}\left(\frac{5\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}} \quad (3)$$

Just carry any $\sqrt{2}$ factors along as scalars throughout your derivations. Show all work.
NOTE 2: Using concepts learned in class, this problem should only take a few lines, with the most work involving what the frequency response of the filter is doing at the frequency $\omega = \frac{2\pi}{8} = \frac{\pi}{4}$.
NOTE 3: There won't be a lot of points for brute force calculations that do not show what you learned in class for Exam 3.

From what we learned in class, this is the "lucky" case where the time-domain aliasing the transient points at the end into the transient points at the beginning yields "good" points.

Answer: $y_8[n] = \left|H\left(\frac{2\pi}{8}\right)\right| \sin\left(\frac{2\pi}{8}n + \angle H\left(\frac{2\pi}{8}\right)\right)\{u[n] - u[n-8]\}$

So, we need to determine $H(\omega)$ and find the value of $H\left(\frac{2\pi}{8}\right) = H\left(\frac{\pi}{4}\right)$

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$$r[n]=u[n]-u[n-4] \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{3}{2}\omega} = R(\omega)$$

$$H(\omega) = \frac{1}{2} R\left(\omega-\frac{\pi}{4}\right) + \frac{1}{2} R\left(\omega+\frac{\pi}{4}\right)$$

$$= \underbrace{\frac{1}{2}\frac{\sin\left(2\left(\omega-\frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega-\frac{\pi}{4}\right)\right)} e^{-j\frac{3}{2}\left(\omega-\frac{\pi}{4}\right)}}_{\text{at } \omega=\frac{\pi}{4},\text{ this is equal to } 4\cdot\frac{1}{2}=2} + \underbrace{\frac{1}{2}\frac{\sin\left(2\left(\omega+\frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{4}\right)\right)} e^{-j\frac{3}{2}\left(\omega+\frac{\pi}{4}\right)}}_{\text{at } \omega=\frac{\pi}{4},\text{ this is zero}}$$

since: $\sin\left(2\frac{\pi}{2}\right)=\sin(\pi)=0$

Thus, $H\left(\frac{\pi}{4}\right) = 4\cdot\frac{1}{2} = 2$

$= H_8(1)$

Thus, final answer:

$$y_8[n] = 2\sin\left(\frac{2\pi}{8}n\right)\left\{u[n]-u[n-8]\right\}$$

$$= 2\left\{\underset{\substack{\uparrow \\ n=0}}{0}, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}\right\}$$

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Problem 3. NOTE: the trigonometric values in Eqn (3) in Problem 2 will be useful in part (c) below as well; just carry any $\sqrt{2}$ factors along as a scalar. Consider a causal FIR filter of length $M = 10$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \left\{ \frac{\sin\left[\frac{\pi}{4}(n+\ell 10)\right]}{\pi(n+\ell 10)} + \frac{\sin\left[\frac{\pi}{2}(n+\ell 10)\right]}{\pi(n+\ell 10)} + \frac{\sin\left[\frac{3\pi}{4}(n+\ell 10)\right]}{\pi(n+\ell 10)} \right\} \{u[n] - u[n-10]\}$$

(a) Determine the 10-pt DFT of $h_p[n]$, denoted $H_{10}(k)$, for $0 \le k \le 9$. Write your answer in sequence form to indicate the numerical values of $H_{10}(k), k = 0, 1, ..., 9$.

(b) Consider the sequence $x[n]$ of length $L = 10$ below, equal to a sum of 10 finite-length sinewaves, each having a different amplitude as indicated below.

$$x[n] = \sum_{k=0}^{9} (10-k) e^{jk\frac{2\pi}{10}n} \{u[n] - u[n-10]\}$$

$y_{10}[n]$ is formed by computing $X_{10}(k)$ as a 10-pt DFT of $x[n]$, $H_{10}(k)$ as a 10-pt DFT of $h[n]$ and, finally, then $y_{10}[n]$ is computed as the 10-pt inverse DFT of $Y_{10}(k) = X_{10}(k)H_{10}(k)$. Express the result $y_{10}[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above. **Clearly indicate which of the 10 equi-spaced frequencies are nulled out, i.e., are not present in** $y_{10}[n]$.

Next, consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = 2\sqrt{2}\, j \sum_{\ell=-\infty}^{\infty} \left\{ \frac{\sin\left[\pi(n-1+\ell 8)\right]}{\pi(n-1+\ell 8)} - \frac{\sin\left[\pi(n+1+\ell 8)\right]}{\pi(n+1+\ell 8)} \right\} \{u[n] - u[n-8]\}$$

(c) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \le k \le 7$. Write your answer in sequence form to indicate the numerical values of $H_8(k), k = 0, 1, ..., 7$.

(d) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves as indicated below.

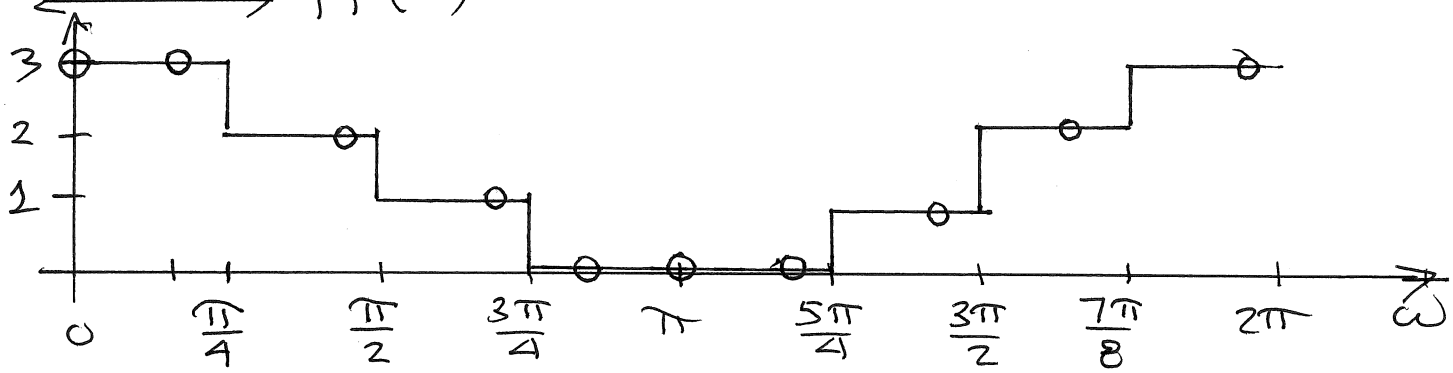
$$x[n] = \sum_{k=0}^{7} e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as a 8-pt DFT of $x[n]$, $H_8(k)$ as a 8-pt DFT of $h[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above. **Clearly indicate which of the 8 equi-spaced frequencies are nulled out, i.e., are not present in** $y_8[n]$.

(a) *This page left intentionally blank for student work.*

$$h[n] = \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} + \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} + \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n}$$

$$\xleftrightarrow{\text{DTFT}} H(\omega)$$



sampled at $\omega_k = k\frac{2\pi}{10} = k\frac{\pi}{5}$ $\quad k = 0, 1, \ldots, 9$

$$= 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}$$

$$\Rightarrow \{3, 3, 2, 1, 0, 0, 0, 1, 2, 3\} = H_{10}(k)$$

$k=0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$

(b) $$y_{10}[n] = \sum_{k=0}^{9} H_{10}(k)\,(10-k)\,e^{jk\frac{2\pi}{10}n}\,\{u[n]-u[n-10]\}$$

$H_{10}(4) = 0 \Rightarrow 4\,\frac{2\pi}{10} = \frac{4\pi}{5} \Rightarrow$ nulled out

$H_{10}(5) = 0 \Rightarrow 5\,\frac{2\pi}{10} = \pi \Rightarrow$ nulled out

$H_{10}(6) = 0 \Rightarrow 6\,\frac{2\pi}{10} = \frac{6\pi}{5} \Rightarrow$ nulled out

same as $\frac{6\pi}{5} - \frac{10\pi}{5} = -\frac{4\pi}{5}$

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(c) $h[n] = \left\{ \frac{\sin(\pi(n-1))}{\pi(n-1)} - \frac{\sin(\pi(n+1))}{\pi(n+1)} \right\} 2\sqrt{2}j$

$$\xleftrightarrow{DTFT} 2j\sqrt{2}\left\{e^{-jw} - e^{jw}\right\}\frac{2j}{2j} = -4\sqrt{2}(-1)\sin(w)$$

$$= 4\sqrt{2}\sin(w)$$

Sampled at:

$w_k = k\frac{2\pi}{8} = k\frac{\pi}{4}$

$k = 0, 1, \ldots, 7$

w:	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$H(w)$:	0	4	$4\sqrt{2}$	4	0	-4	$-4\sqrt{2}$	-4
k:	0	1	2	3	4	5	6	7

$$H_8(k) = \{\underset{\substack{\uparrow \\ k=0}}{0}, 4, 4\sqrt{2}, 4, 0, -4, -4\sqrt{2}, -4\}$$

(d) $y_8[n] = \sum_{k=0}^{7} H_8(k)\, e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$

$H_8(0) = 0 \Rightarrow w = 0$ is nulled out

$H_8(4) = 0 \Rightarrow w = \pi$ is nulled out

$= 4\frac{2\pi}{8}$