Exam 3 Solution Fall 2009

Problem 1

(a) Sequence $x[n]$ is of length $L=12$

(i) $N=16 > L=12$, so no time-domain aliasing

$$X_r(\omega) = X(\omega) = \frac{\sin\left(\frac{12}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(12-1)}{2}\omega}$$

(ii) $N=12 = L=12 \Rightarrow$ no time-domain aliasing

$$X_r(\omega) = X(\omega) = \frac{\sin(6\omega)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{11}{2}\omega}$$

(iii) $N=8 < L=12 \Rightarrow$ time-domain aliasing

Thus, $x_8[n] = x[n] + x[n+8]$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

$$= 2\{u[n] - u[n-3]\} + \{u[n'] - u[n'-3]\}$$

$n' = n-4$

$$X_r(\omega) = \{2 + e^{-j4\omega}\} \frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(4-1)}{2}\omega}$$

$$= \{2 + e^{-j4\omega}\} \frac{\sin(2\omega)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j3\omega}$$

Exam 3 Solution

Prob. 1 (b): Since $N=16 = L=16$, no time-domain aliasing $\Rightarrow X_r(\omega) = X(\omega)$

Since: $X_r(\omega) = X(\omega) =$ (see ~~below~~) see next page

then: $X_r\left(2\pi \frac{k_0}{N} n\right) = X\left(2\pi \frac{k_0}{N} n\right) = X_N(k)$

where $N=16$. $\frac{\pi}{4} = \frac{2\pi k_0}{16} \Rightarrow k_0 = 2$

$X_N(k) =$

$$\frac{16}{2}\delta[k-2] + \frac{16}{2}\delta[k-(16-k_0)]$$

$$= 8\,\delta[k-2] + 8\,\delta[k-14]$$

(ii) $\frac{\pi}{8} = 2\pi\frac{k}{16} \Rightarrow k=1 \quad X_N(1) = 0$

$$X_r\left(\frac{\pi}{8}\right) = 0$$

(iii) $\frac{\pi}{4} = 2\pi\frac{k}{16} \Rightarrow k=2 \quad X_N(2) = 8$

$$X_r\left(\frac{\pi}{4}\right) = 8$$

(iv) $\frac{7\pi}{4} = 2\pi\frac{k}{16} \Rightarrow k=14 \quad X_N(14) = 8$

$$X_r\left(\frac{7\pi}{4}\right) = 8$$

Answer to Prob. 1 (b)-(i):

$$X_r(\omega) = X(\omega) = \frac{1}{2}\,\frac{\sin\left(\frac{16}{2}\left(\omega-\frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega-\frac{\pi}{4}\right)\right)}\, e^{-j\frac{(16-1)}{2}\left(\omega-\frac{\pi}{4}\right)}$$

$$+\ \frac{1}{2}\,\frac{\sin\left(\frac{16}{2}\left(\omega+\frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{4}\right)\right)}\, e^{-j\frac{(16-1)}{2}\left(\omega+\frac{\pi}{4}\right)}$$

Problem 2. This problem uses the basic DFT pair:

$$e^{j2\pi\frac{k_0}{N}n}\{u[n]-u[n-N]\} \overset{DFT}{\underset{N}{\longleftrightarrow}} N\delta(k-k_0)$$

$k_0 \in \{0, 1, \ldots, N-1\}$

Consider $h[n]$ an FIR filter of length $M<N$

$$x[n] = e^{j2\pi\frac{k_0}{N}n} \overset{DFT}{\underset{N}{\longleftrightarrow}} X_N(k) = N\delta(k-k_0) \qquad h[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} H_N(k)$$

$$y_N[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} Y_N(k) = X_N(k)H_N(k)$$

$$= N\delta(k-k_0)H_N(k)$$

$$= NH_N(k_0)\delta(k-k_0)$$

$$y_N[n] \overset{DFT}{\underset{N}{\longleftrightarrow}}$$

$$= H_N(k_0)e^{j2\pi\frac{k_0}{N}n}\{u[n]-u[n-N]\}$$

$N=8$:

$$x[n] = \left\{ 3e^{j2\pi\frac{(0)}{8}n} + \frac{1}{2}e^{j2\pi\frac{(2)}{8}n} + \frac{1}{2}e^{-j2\pi\frac{(2)}{8}n} + 2e^{j\frac{2\pi(4)n}{8}} \right\}\{u[n]-u[n-8]\}$$

Note: $e^{j\pi n} = \cos(\pi n) = e^{j2\pi\frac{(4)}{8}n}$

$$\frac{1}{2}e^{-j2\pi\frac{(2)}{8}n} = \frac{1}{2}e^{j2\pi\frac{(8-2)}{8}n} = \frac{1}{2}e^{j2\pi\frac{6}{8}n}$$

$$3e^{j2\pi\frac{(0)}{8}n} \Rightarrow 3H_8(0)\,e^{j2\pi\frac{(0)}{8}n}$$

$$\frac{1}{2}e^{j2\pi\frac{(2)}{8}n} \Rightarrow \frac{1}{2}H_8(2)\,e^{j2\pi\frac{(2)}{8}n}$$

$$2e^{j2\pi\frac{(4)}{8}n} \Rightarrow 2H_8(4)\,e^{j2\pi\frac{(4)}{8}n}$$

$$\frac{1}{2}e^{j2\pi\frac{(6)}{8}n} \Rightarrow \frac{1}{2}H_8(6)\,e^{j2\pi\frac{(6)}{8}n}$$

Given: $\{1,-2,1\} = \{1,-1\} * \{1,-1\}$

$$\{1,-1\} \xleftrightarrow{DTFT} 2j\sin\left(\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}}$$

$$H(\omega) = \left\{2j\sin\left(\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}}\right\}^2$$

$$= -4\sin^2\left(\frac{\omega}{2}\right)e^{-j\omega}$$

$$H_8(k) = H(\omega)\Big|_{\omega = 2\pi\frac{k}{8}}$$

$$= -4\sin^2\left(\frac{1}{2}\,2\pi\frac{k}{8}\right) e^{-j\,2\pi\frac{k}{8}}$$

$$= -4\sin^2\left(k\frac{\pi}{8}\right) e^{-j\,2\pi\frac{k}{8}}$$

$$H_8(0) = 0$$

$$H_8(2) = -4\sin^2\left(\frac{\pi}{4}\right) e^{-j\,2\pi\frac{(2)}{8}}$$

$$= -4\left(\frac{1}{\sqrt{2}}\right)^2 e^{-j\frac{\pi}{2}}$$

$$= -2(-j) = 2j$$

$$H_8(4) = -4\sin^2\left(4\frac{\pi}{8}\right) e^{-j\,2\pi\frac{(4)}{8}}$$

$$= -4(1)^2(-1)$$

$$= 4$$

$$H_8(6) = H_8^*(2) = -2j$$

Thus, the answer is:

$$y_8[n] = \cancel{3(0)\, e^{j 2\pi \frac{(0)}{8} n}} \rightarrow \emptyset$$

$$+ \frac{1}{2}(2j)\, e^{j 2\pi \frac{(2)}{8} n}$$

$$+ 2(4)\, e^{j 2\pi \frac{(4)}{8} n}$$

$$+ \frac{1}{2}(-2j)\, e^{j 2\pi \frac{(6)}{8} n}$$

$$y_8[n] = \left\{ -2 \sin\left(\frac{\pi}{2} n\right) + 8 \cos(\pi n) \right\} \left(u[n] - u[n-8]\right)$$

Alternative solution:

$$x[n] = \{3 + \cos(\tfrac{\pi}{2}n) + 2\cos(\pi n)\}[u[n]-u[n-8]]$$

$$= \{3,3,3,3,3,3,3,3\}$$

$$+ \{1,0,-1,0,1,0,-1,0\}$$

$$+ \{2,-2,2,-2,2,-2,2,-2\}$$

$$= \{6,1,4,1,6,1,4,1\}$$

linear convolution: $y[n] = x[n] * h[n]$

$$= x[n] * \{1,-2,1\}$$

$$= \{6,-11,8,-6,8,-10,8,-6,2,1\}$$

it length $8+3-1 = 10$

Since using 8-pt DFT's, $10-8=2$ points at end aliased into two points at beginning:

$$\begin{array}{cc} 6, & -11 \\ 2, & 1 \\ \hline 8, & -10 \end{array}$$

Thus: $y_8[n] = \{8,-10,8,-6,8,-10,8,-6\}$

Another alternative solution:
Use superposition (linearity)

$$\{3,3,3,3,3,3,3,3\} * \{1,-1\} * \{1,-1\}$$
$$+ \{1,0,-1,0,1,0,-1,0\} * \{1,-1\} * \{1,-1\}$$
$$+ \{2,-2,2,-2,2,-2,2,-2\} * \{1,-1\} * \{1,-1\}$$
$$= \{6,-11,8,-6,8,-10,8,-6,2,1\}$$

Then, again alias last 2 points into first 2 points

$$y_8[n] = \{8,-10,8,-6,8,-10,8,-6\}$$