

SOLUTION

NAME: _____ 2016
Digital Signal Processing I Exam 3 Fall 2016
Session 40 2 Dec. 2016

Cover Sheet

Test Duration: 60 minutes.

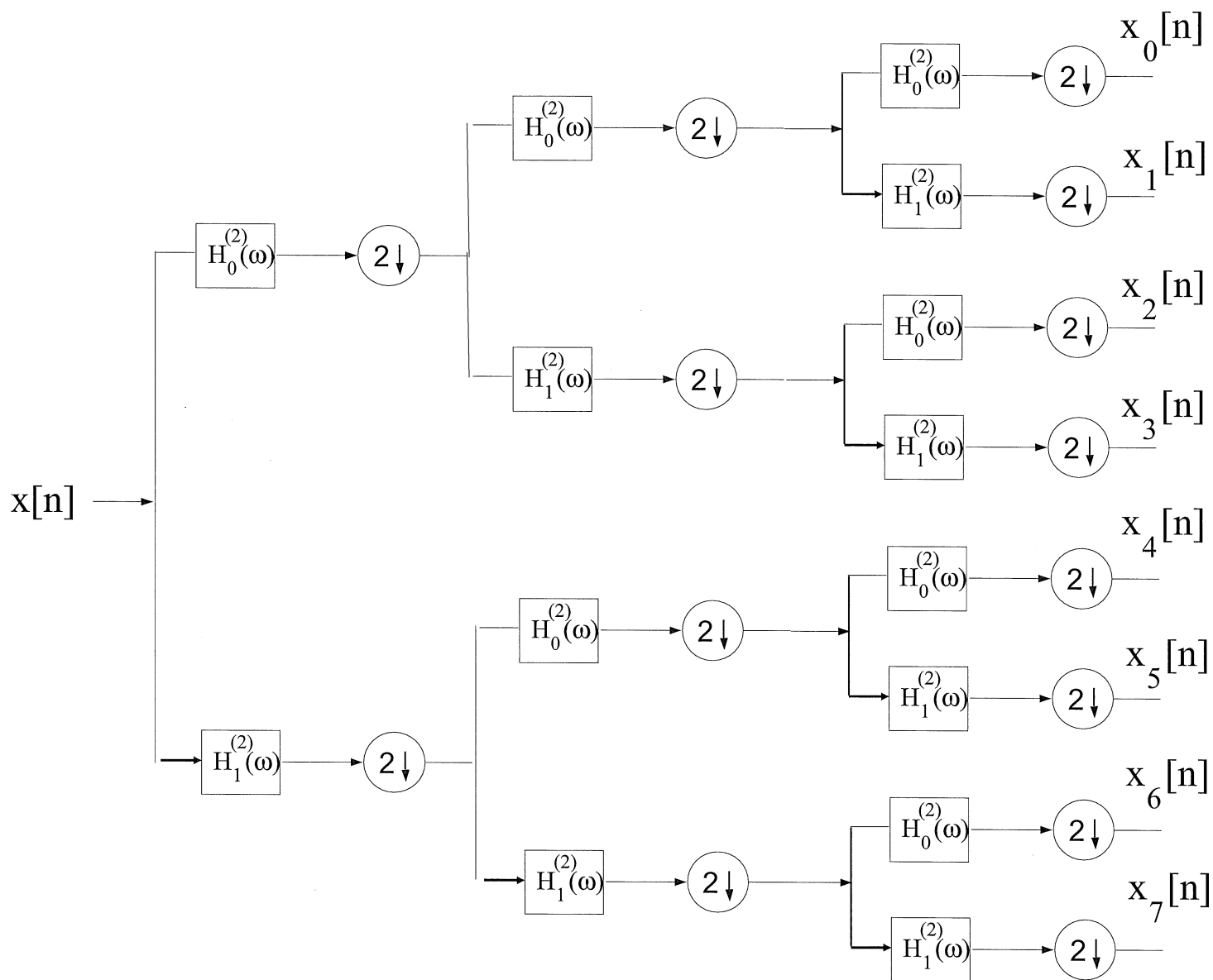
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed

Calculators NOT allowed.

This test contains **THREE** problems.

All work should be done on the blank pages provided.

Your answer to each part of the exam should be clearly labeled.

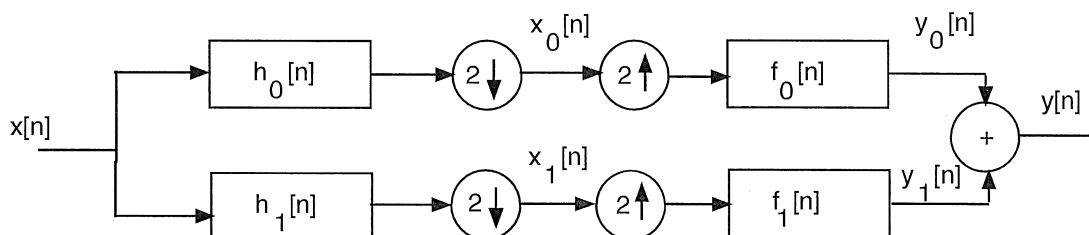


Problem 1. In the system below, the two analysis filters, $h_0[n]$ and $h_1[n]$, and the two synthesis filters, $f_0[n]$ and $f_1[n]$, form a Quadrature Mirror Filter (QMF). Specifically,

$$h_1[n] = (-1)^n h_0[n] \quad f_0[n] = h_0[n] \quad f_1[n] = -h_1[n]$$

The lowpass (halfband) filter $h_0[n]$ employed is the following 2-tap filter:

$$h_0[n] = \delta[n] + j\delta[n-1] = \{1, j\}$$



Determine mathematically (include as much detail as possible) if the lowpass half-band filter above satisfies the condition required for Perfect Reconstruction. Be sure to clearly state what that condition is (don't need to rederive it) and then show whether it is satisfied with the filter $h_0[n]$, showing as much detail as possible.

$$\frac{1}{2} \{ H_0^2(\omega) - H_0^2(\omega - \pi) \} = c e^{-jn_0\omega}$$

$$\begin{aligned} H_0^2(\omega) &= (1 + je^{-j\omega})(1 + je^{-j\omega}) \\ &= 1 - e^{-j2\omega} + je^{-j\omega} \cdot 2 \end{aligned}$$

$$\begin{aligned} H_0^2(\omega - \pi) &= (1 + je^{-j(\omega - \pi)})^2 = (1 - je^{-j\omega})^2 \\ &= 1 - 2je^{-j\omega} + e^{-j2\omega} \end{aligned}$$

$$H_0^2(\omega) - H_0^2(\omega - \pi) = 4je^{-j\omega}$$

$$\frac{1}{2} \{ H_0^2(\omega) - H_0^2(\omega - \pi) \} = \underbrace{2j}_c e^{-j\omega} \quad n_0 = 1$$

$$H_0^{(2)}(\omega) 1 + e^{-j\omega} = 2 e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right) \quad 1 - e^{-j\omega} = 2j e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right)$$

Problem 2. In Matlab Homework 2, you used Noble's Identities to convert the Tree-Structured subbander on the previous page to the regular maximally decimated subbander, obtaining impulse responses of the form:

$$h_\ell[n] = h_a^{(2)}[n] * \sum_{k=-\infty}^{\infty} h_b^{(2)}[k] \delta[n-2k] * \sum_{k=-\infty}^{\infty} h_c^{(2)}[k] \delta[n-4k] \quad \ell = 0, 1, \dots, 7$$

where $a \in \{0, 1\}$, $b \in \{0, 1\}$, and $c \in \{0, 1\}$, with $h_0^{(2)}[n] = \{1, 1\}$ and $h_1^{(2)}[n] = \{1, -1\}$ in the two-channel QMF. You are required to fill in the TWO tables below. I have already filled in the last entry of each table to show you the kind of expressions that I am looking for.

$$\begin{aligned} h_0[n] &= \{1 & 1 & 1 & 1 & 1 & 1 & 1\} \\ h_1[n] &= \{1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\} \\ h_2[n] &= \{1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\} \\ h_3[n] &= \{1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\} \\ h_4[n] &= \{1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\} \\ h_5[n] &= \{1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\} \\ h_6[n] &= \{1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\} \\ h_7[n] &= \{1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\} \end{aligned} \quad (1)$$

$$\begin{aligned} H_0(\omega) &= H_0^{(2)}(\omega) H_0^{(2)}(2\omega) H_0^{(2)}(4\omega) = 8 e^{-j3.5\omega} \cos\left(\frac{\omega}{2}\right) \cos(\omega) \cos(2\omega) \\ H_1(\omega) &= H_0^{(2)}(\omega) H_0^{(2)}(2\omega) H_1^{(2)}(4\omega) = j8 e^{-j3.5\omega} \cos\left(\frac{\omega}{2}\right) \cos(\omega) \sin(2\omega) \\ H_2(\omega) &= H_0^{(2)}(\omega) H_1^{(2)}(2\omega) H_0^{(2)}(4\omega) = -j8 e^{-j3.5\omega} \cos\left(\frac{\omega}{2}\right) \sin(\omega) \cos(2\omega) \\ H_3(\omega) &= H_0^{(2)}(\omega) H_1^{(2)}(2\omega) H_1^{(2)}(4\omega) = -8 e^{-j3.5\omega} \cos\left(\frac{\omega}{2}\right) \sin(\omega) \sin(2\omega) \\ H_4(\omega) &= H_1^{(2)}(\omega) H_0^{(2)}(2\omega) H_0^{(2)}(4\omega) = j8 e^{-j3.5\omega} \sin\left(\frac{\omega}{2}\right) \cos(\omega) \cos(2\omega) \\ H_5(\omega) &= H_1^{(2)}(\omega) H_0^{(2)}(2\omega) H_1^{(2)}(4\omega) = -j8 e^{-j3.5\omega} \sin\left(\frac{\omega}{2}\right) \cos(\omega) \sin(2\omega) \\ H_6(\omega) &= H_1^{(2)}(\omega) H_1^{(2)}(2\omega) H_0^{(2)}(4\omega) = -8 e^{-j3.5\omega} \sin\left(\frac{\omega}{2}\right) \sin(\omega) \cos(2\omega) \\ H_7(\omega) &= H_1^{(2)}(\omega) H_1^{(2)}(2\omega) H_1^{(2)}(4\omega) = 8 e^{-j3.5\omega} \sin\left(\frac{\omega}{2}\right) \sin(\omega) \sin(2\omega) \end{aligned}$$

$$H_7(\omega) = 2j \sin\left\{\frac{\omega}{2}\right\} e^{-j\frac{\omega}{2}} 2j \sin\{\omega\} e^{-j\omega} 2j \sin\{2\omega\} e^{-j2\omega}$$

$$\begin{aligned} \frac{1}{2} (0 + \pi/8) &\rightarrow H_0(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } \pi/16 \\ \frac{1}{2} (\pi/8 + 2\pi/8) &\rightarrow H_1(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 3\pi/16 \\ \frac{1}{2} (3\pi/8 + 4\pi/8) &\rightarrow H_2(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 7\pi/16 \\ \frac{1}{2} (5\pi/8 + 6\pi/8) &\rightarrow H_3(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 5\pi/16 \\ 0.5(2\pi/8 + 3\pi/8) &\rightarrow H_4(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 5\pi/16 \\ 0.5(7\pi/8 + 8\pi/8) &\rightarrow H_5(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 13\pi/16 \\ 0.5(6\pi/8 + 7\pi/8) &\rightarrow H_6(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 9\pi/16 \\ 0.5(4\pi/8 + 5\pi/8) &\rightarrow H_7(\omega) \quad (\text{passband center}) \text{ frequency at which max occurs: } 11\pi/16 \\ 0.5(5\pi/8 + 6\pi/8) &\rightarrow \end{aligned} \quad (3)$$

see soln to Prob. 2 from Fall 2015

middle of corresponding frequency bands

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Problem 3. Consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin \left[\pi \left(n - \frac{3}{4} + \ell 8 \right) \right]}{\pi \left(n - \frac{3}{4} + \ell 8 \right)} \{u[n] - u[n-8]\}$$

- (a) Determine the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \leq k \leq 7$. EITHER write an expression for $H_8(k)$ OR list the numerical values: $H_8(0) = ?$, $H_8(1) = ?$, $H_8(2) = ?$, $H_8(3) = ?$, $H_8(4) = ?$, $H_8(5) = ?$, $H_8(6) = ?$, $H_8(7) = ?$.
- (b) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves, each having a different amplitude as indicated below.

$$x[n] = \sum_{k=0}^7 (8-k) e^{jk \frac{2\pi}{8} n} \{u[n] - u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as a 8-pt DFT of $x[n]$, $H_8(k)$ as a 8-pt DFT of $h[n]$ and, finally, then $y_8[n]$ is computed as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

Next, consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \left\{ \frac{\sin \left[\frac{\pi}{2} (n-1 + \ell 8) \right]}{\pi (n-1 + \ell 8)} + \frac{\sin \left[\frac{\pi}{2} (n+1 + \ell 8) \right]}{\pi (n+1 + \ell 8)} \right\} \{u[n] - u[n-8]\}$$

- (c) Determine all 10 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \leq k \leq 9$. List the values clearly: $H_8(0) = ?$, $H_8(1) = ?$, $H_8(2) = ?$, $H_8(3) = ?$, $H_8(4) = ?$, $H_8(5) = ?$, $H_8(6) = ?$, $H_8(7) = ?$.
- (d) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves, each having a different amplitude as indicated below.

$$x[n] = \sum_{k=0}^7 e^{jk \frac{\pi}{4}} e^{jk \frac{2\pi}{8} n} \{u[n] - u[n-8]\}$$

$y_8(n) = [n]$ is formed by computing $X_8(k)$ as a 8-pt DFT of $x[n]$, $H_8(k)$ as a 8-pt DFT of $h[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

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(a) $l=0$: $\frac{\sin(\pi(n-\frac{3}{4}))}{\pi(n-\frac{3}{4})} \xleftrightarrow{\text{DTFT}} e^{-j\frac{3}{4}\omega}$ for $-\pi < \omega < \pi$

at $\omega = \pi^- \Rightarrow e^{-j\frac{3}{4}\pi}$

at $\omega = \pi^+ \Rightarrow$ same as $\omega = -\pi^+ = e^{j\frac{3}{4}\pi}$ $k\frac{2\pi}{8} = \pi \quad \underline{\underline{k=4}}$

$e^{-j\frac{3}{4}\pi} \rightarrow e^{j\frac{3}{4}\pi} \uparrow e^{j\frac{3}{2}\pi} = -j$

Thus, for $\pi < \omega < 2\pi$, $H_0(\omega) = +j e^{-j\frac{3}{4}\omega}$

$H_8(k) = e^{-j\frac{3}{4}k\frac{2\pi}{8}} = e^{-jk\frac{3\pi}{16}} \quad k=0,1,2,3$

$H_8(k) = 0 \quad \text{for } k=4$

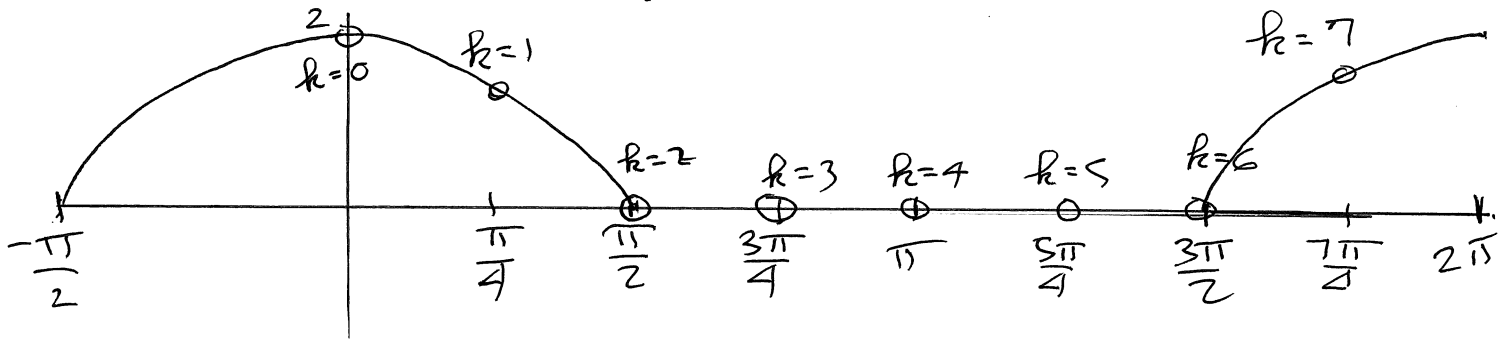
$H_8(k) = \underbrace{-j}_{e^{j\frac{3\pi}{2}}} e^{-j\frac{3}{4}k\frac{2\pi}{8}} = \underbrace{-j}_{e^{j\frac{3\pi}{2}}} e^{-jk\frac{3\pi}{16}} \quad k=5,6,7$

(b) $y_8[n] = \sum_{k=0}^7 H_8(k) (8-k) e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$

sum of 8 "lucky" cases

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$$\begin{aligned}
 (c) \quad l=0: \quad & \frac{\sin\left(\frac{\pi}{2}(n-1)\right)}{\pi(n-1)} + \frac{\sin\left(\frac{\pi}{2}(n+1)\right)}{\pi(n+1)} \xleftrightarrow{\text{DTFT}} \\
 & \text{rect}\left(\frac{\omega}{2 \cdot \frac{\pi}{2}}\right) \left\{ e^{-j\omega} + e^{j\omega} \right\} \quad \text{for } -\pi < \omega < \pi \\
 & \text{periodic @} \\
 & \text{period} = 2\pi \\
 & = 2\cos(\omega) \text{rect}\left(\frac{\omega}{\pi}\right)
 \end{aligned}$$



$$k \frac{2\pi}{8} = k \frac{\pi}{4} \Rightarrow$$

$$\begin{aligned}
 X_8(k) &= \left\{ 2, \frac{2}{\sqrt{2}}, 0, 0, 0, 0, 0, \frac{2}{\sqrt{2}} \right\} \\
 &= \left\{ 2, \sqrt{2}, 0, 0, 0, 0, 0, \sqrt{2} \right\} \\
 &\quad \uparrow \\
 &\quad k=0
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad x_8[n] &= \sum_{k=0}^7 e^{j k \frac{\pi}{4}} H_8(k) e^{-j k \frac{2\pi}{8} n} \\
 &= 2 + \sqrt{2} e^{j\left(\frac{2\pi}{8}n + \frac{\pi}{4}\right)} + \sqrt{2} e^{j\left(\frac{7\pi}{8}n + \frac{7\pi}{4}\right)}
 \end{aligned}$$