

SOLUTION

NAME: 2015
Digital Signal Processing I Exam 3 Fall 2015
Session 40 4 Dec. 2015

Cover Sheet

Test Duration: 60 minutes.

Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed

Calculators NOT allowed.

This test contains **THREE** problems.

All work should be done on the blank pages provided.

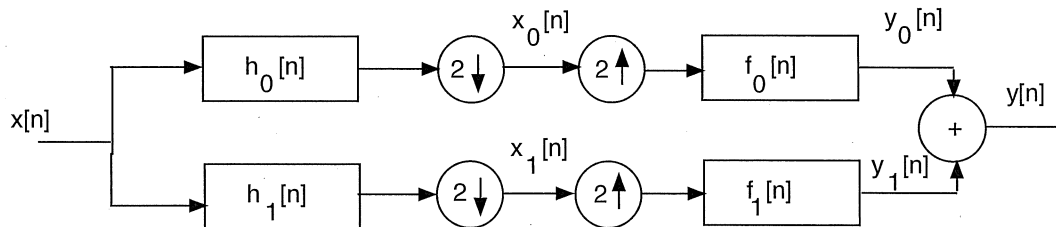
Your answer to each part of the exam should be clearly labeled.

Problem 1. In the system below, the two analysis filters, $h_0[n]$ and $h_1[n]$, and the two synthesis filters, $f_0[n]$ and $f_1[n]$, form a Quadrature Mirror Filter (QMF). Specifically,

$$h_1[n] = (-1)^n h_0[n] \quad f_0[n] = h_0[n] \quad f_1[n] = -h_1[n]$$

The lowpass (halfband) filter $h_0[n]$ employed is the following 2-tap filter:

$$h_0[n] = \delta[n] + \delta[n-1] = \{1, 1\}$$



Determine mathematically (include as much detail as possible) if the lowpass half-band filter above satisfies the condition required for Perfect Reconstruction. Be sure to clearly state what that condition is (don't need to rederive it) and then show whether it is satisfied with the filter $h_0[n]$, showing as much detail as possible.

QMF guarantees alias-free.

Distortionless requirement:

$$H_0^2(\omega) - H_0^2(\omega - \pi) = c e^{-j\omega}$$

$$H_0(\omega) = 1 + e^{-j\omega}$$

Thus:

$$(1 + e^{-j\omega})^2 - (1 + e^{-j(\omega - \pi)})^2 = ?$$

$$(1 + 2e^{-j\omega} + e^{-j2\omega}) - (1 - 2e^{-j\omega} + e^{-j2\omega})$$

$$= 4e^{-j\omega} \quad \checkmark$$

Alternatively, use half-angle trick

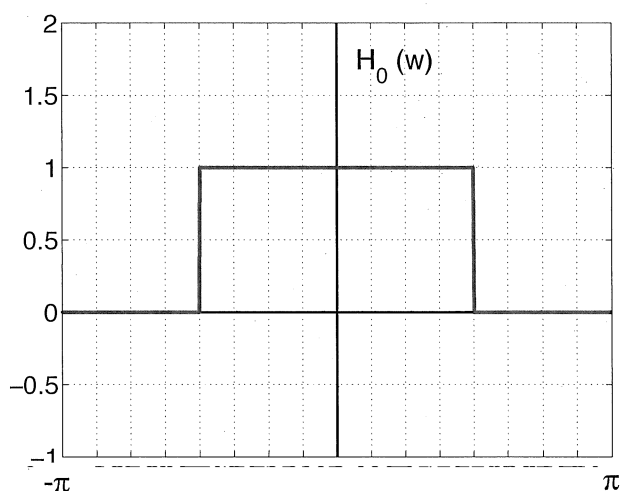
$$H_0(\omega) = 2 e^{j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$

$$H_1(\omega) = H_0(\omega - \pi) = 2j e^{j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right)$$

Problem 2. In Matlab Homework 2, you used Noble's Identities to convert the Tree-Structured subbander on the previous page to the regular maximally decimated subbander, obtaining products of the form:

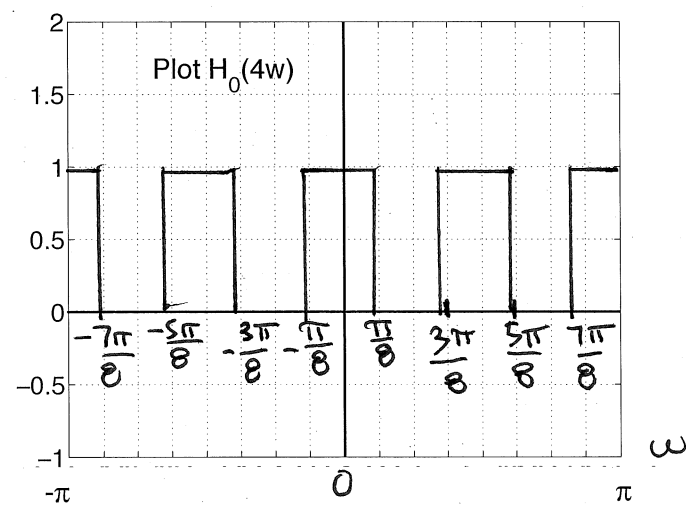
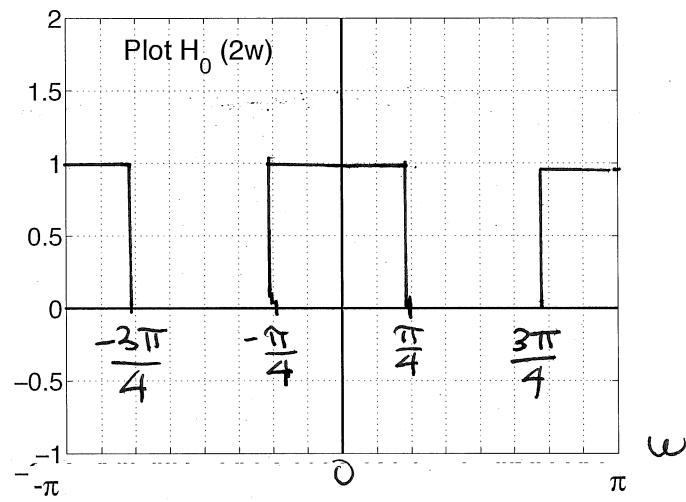
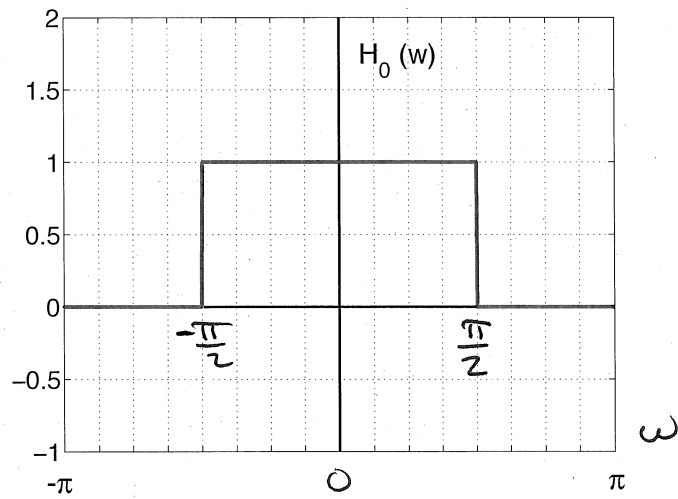
$$H_k(\omega) = H_\ell^{(2)}(\omega) H_m^{(2)}(2\omega) H_n^{(2)}(4\omega) \quad k = 0, 1, \dots, 7$$

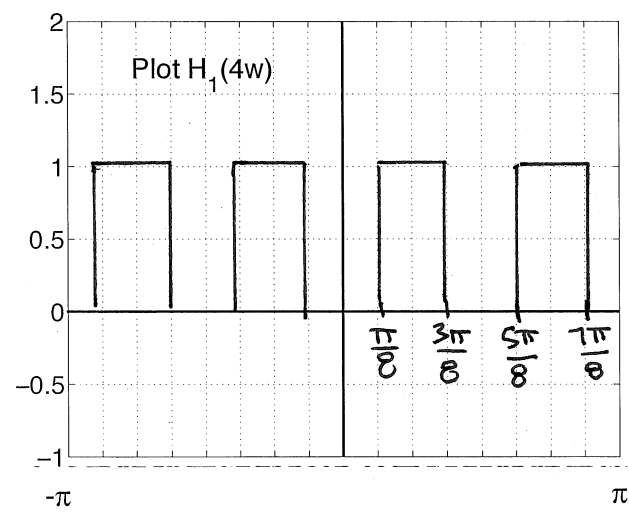
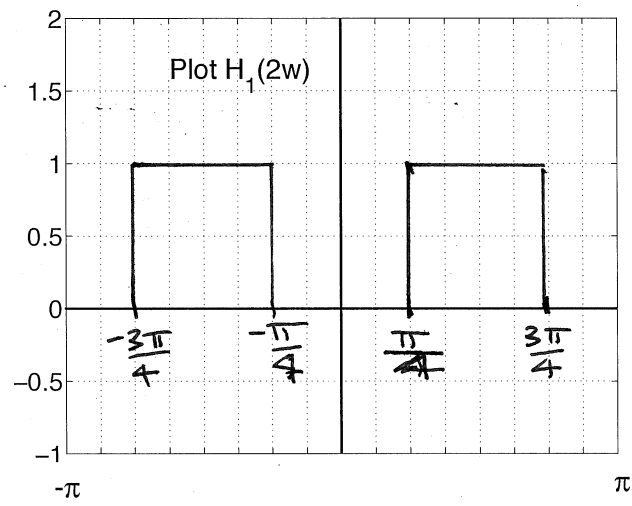
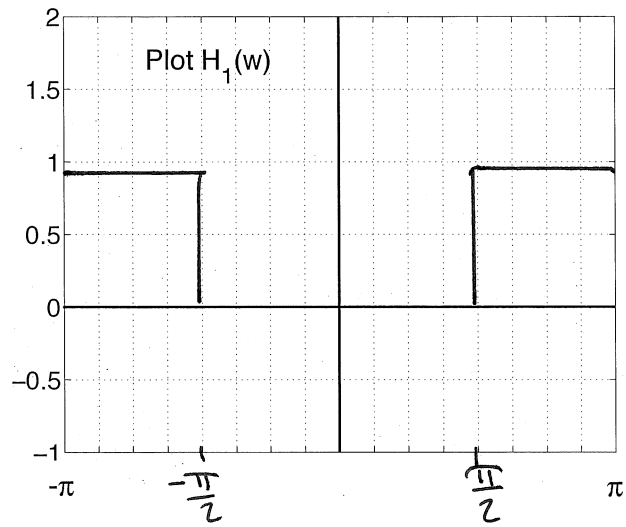
where $\ell \in \{0, 1\}$, $m \in \{0, 1\}$, and $n \in \{0, 1\}$, with $H_0^{(2)}(\omega)$ denoting the lowpass halfband filter below for the two-channel QMF and $H_1^{(2)}(\omega) = H_0^{(2)}(\omega - \pi)$. For this problem, $H_0^{(2)}(\omega)$ is the ideal lowpass filter below (note: a DTFT is always periodic with period 2π .)



You are required to fill in the table below. **FIRST** you should plot $H_0^{(2)}(2\omega)$ and $H_0^{(2)}(4\omega)$ on the next page, and then plot $H_1^{(2)}(\omega)$, $H_0^{(2)}(2\omega)$ and $H_0^{(2)}(4\omega)$ on the next page after that, in the space provided. In each plot, the abscissa range is from $-\pi$ to $+\pi$ and there is a tic mark and vertical dashed line at every integer multiple of $\pi/8$. These plots will help you fill in the table. I have already filled in the first entry. Only fill in the positive frequency band that is passed; the filters are real-valued and even-symmetric, so their respective frequency responses are real-valued and even-symmetric.

$(0, \pi/8)$	$H_0^{(2)}(\omega) H_0^{(2)}(2\omega) H_0^{(2)}(4\omega)$	passes: $0 < \omega < \pi/8$
$(\pi/8, 2\pi/8)$	$H_0^{(2)}(\omega) H_0^{(2)}(2\omega) H_1^{(2)}(4\omega)$	passes: $\pi/8 < \omega < \pi/4$
$(3\pi/8, 4\pi/8)$	$H_0^{(2)}(\omega) H_1^{(2)}(2\omega) H_0^{(2)}(4\omega)$	passes: $3\pi/8 < \omega < \pi/2$
$(4\pi/8, 5\pi/8)$	$H_0^{(2)}(\omega) H_1^{(2)}(2\omega) H_1^{(2)}(4\omega)$	passes: $\pi/4 < \omega < 3\pi/8$ (1)
$(5\pi/8, 6\pi/8)$	$H_1^{(2)}(\omega) H_0^{(2)}(2\omega) H_0^{(2)}(4\omega)$	passes: $5\pi/8 < \omega < 3\pi/4$
$(6\pi/8, 7\pi/8)$	$H_1^{(2)}(\omega) H_0^{(2)}(2\omega) H_1^{(2)}(4\omega)$	passes: $3\pi/4 < \omega < 7\pi/8$
$(7\pi/8, 8\pi/8)$	$H_1^{(2)}(\omega) H_1^{(2)}(2\omega) H_0^{(2)}(4\omega)$	passes: $7\pi/8 < \omega < \pi$
$(\pi, 9\pi/8)$	$H_1^{(2)}(\omega) H_1^{(2)}(2\omega) H_1^{(2)}(4\omega)$	passes: $\pi < \omega < 9\pi/8$





Problem 3. Consider a causal FIR filter of length $M = 10$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin \left[\pi \left(n + \frac{1}{2} + \ell 10 \right) \right]}{\pi \left(n + \frac{1}{2} + \ell 10 \right)} \{u[n] - u[n - 10]\}$$

- (a) Determine the 10-pt DFT of $h_p[n]$, denoted $H_{10}(k)$, for $0 \leq k \leq 9$. EITHER write an expression for $H_{10}(k)$ OR list the numerical values: $H_{10}(0) = ?$, $H_{10}(1) = ?$, $H_{10}(2) = ?$, $H_{10}(3) = ?$, $H_{10}(4) = ?$, $H_{10}(5) = ?$, $H_{10}(6) = ?$, $H_{10}(7) = ?$, $H_{10}(8) = ?$, $H_{10}(9) = ?$.
- (b) Consider the sequence $x[n]$ of length $L = 10$ below, equal to a sum of 10 finite-length sinewaves, each having a different amplitude as indicated below.

$$x[n] = \sum_{k=0}^9 k e^{jk \frac{2\pi}{10} n} \{u[n] - u[n - 10]\}$$

$y_{10}[n]$ is formed by computing $X_{10}(k)$ as a 10-pt DFT of $x[n]$, $H_{10}(k)$ as a 10-pt DFT of $h[n]$ and, finally, then $y_{10}[n]$ is computed as the 10-pt inverse DFT of $Y_{10}(k) = X_{10}(k)H_{10}(k)$. Express the result $y_{10}[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

Next, consider a causal FIR filter of length $M = 10$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \left\{ \frac{\sin \left[\frac{3\pi}{4} (n + \ell 10) \right]}{\pi (n + \ell 10)} + \frac{\sin \left[\frac{\pi}{4} (n + \ell 10) \right]}{\pi (n + \ell 10)} \right\} \{u[n] - u[n - 10]\}$$

- (c) Determine all 10 numerical values of the 10-pt DFT of $h_p[n]$, denoted $H_{10}(k)$, for $0 \leq k \leq 9$. List the values clearly: $H_{10}(0) = ?$, $H_{10}(1) = ?$, $H_{10}(2) = ?$, $H_{10}(3) = ?$, $H_{10}(4) = ?$, $H_{10}(5) = ?$, $H_{10}(6) = ?$, $H_{10}(7) = ?$, $H_{10}(8) = ?$, $H_{10}(9) = ?$.
- (d) Consider the sequence $x[n]$ of length $L = 10$ below, equal to a sum of 10 finite-length sinewaves, each having a different amplitude as indicated below.

$$x[n] = \sum_{k=0}^9 k e^{jk \frac{2\pi}{10} n} \{u[n] - u[n - 10]\}$$

$y_{10}(n) = [n]$ is formed by computing $X_{10}(k)$ as a 10-pt DFT of $x[n]$, $H_{10}(k)$ as a 10-pt DFT of $h[n]$, and then $y_{10}[n]$ as the 10-pt inverse DFT of $Y_{10}(k) = X_{10}(k)H_{10}(k)$. Express the result $y_{10}[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

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$$3a) \quad h[n] = \frac{\sin\left[\pi\left(n + \frac{1}{2}\right)\right]}{\pi\left(n + \frac{1}{2}\right)} = \frac{\sin\left[\frac{\pi}{2}(2n+1)\right]}{\frac{\pi}{2}(2n+1)}$$

$$H(\omega) = \left\{ \frac{1}{2} \sum_{k=0}^1 e^{-j k \frac{2\pi}{2}} \cdot \underbrace{\text{rect}\left(\frac{\omega - k 2\pi}{2\pi}\right)}_{\text{rect}\left(\frac{\omega - k 2\pi}{2\pi}\right)} \right\} e^{j \frac{1}{2} \omega}$$

For $0 < \omega < \pi$:

$$H(\omega) = e^{j \frac{\omega}{2}}$$

For $\pi < \omega < 2\pi$:

$$H(\omega) = -e^{j \frac{\omega}{2}}$$

discontinuity
at $\omega = \pi$

For $0 \leq k \leq 4$:

$$H_N(k) = e^{j \frac{1}{2} k \frac{2\pi}{10}}$$

For $6 \leq k \leq 9$:


$$H_N(k) = \overset{\text{negative sign}}{-} e^{j \frac{1}{2} k \frac{2\pi}{10}}$$

$$\omega = \frac{k 2\pi}{10} = \pi$$

$$k = 5$$

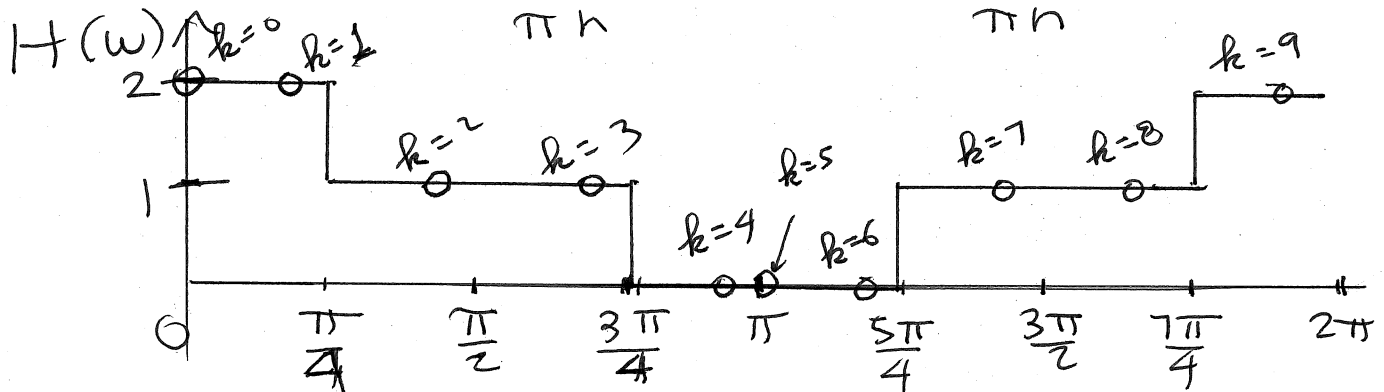
For $k=5$, $H_N(5) = 0$ (or j or $-j$ acceptable)

$$y_{10}(n) = \sum_{k=0}^9 H_N(k) \cdot e^{j k \frac{2\pi}{10} n} \{u[n] - u[n-10]\}$$

3(b) $k=0$ 
defined
in (a)
above

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$$3(c) \quad h[n] = \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n} + \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$$



Sampling at $\omega_k = k \frac{2\pi}{10} \quad k = 0, 1, \dots, 9$
 $= k \frac{\pi}{5}$

$$3(d) \quad y_{10}[n] = \sum_{k=0}^9 k H_N(k) e^{j k \frac{2\pi}{10} n} \{u[n] - u[n-10]\}$$

$$H_N(0) = H_N(1) = 2$$

$$H_N(2) = H_N(3) = 1$$

$$H_N(4) = H_N(5) = H_N(6) = 0$$

$$H_N(7) = H_N(8) = 1$$

$$H_N(9) = 2$$