SOLUTION

NAME: **2014**
Digital Signal Processing I **Exam 3** **Fall 2014**
Session 40 **3 Dec. 2014**

Cover Sheet

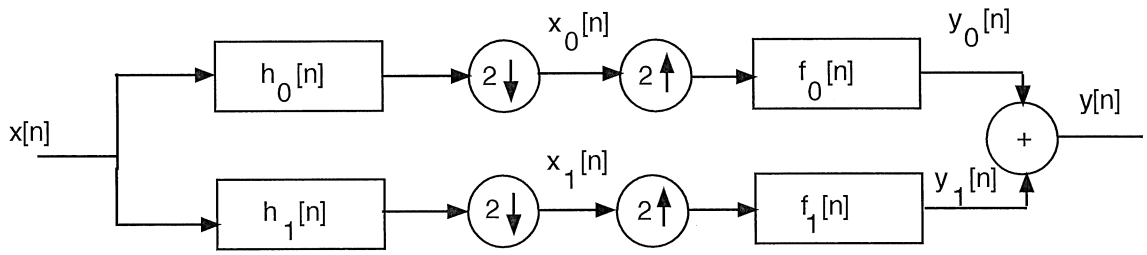
Test Duration: 60 minutes.
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed
Calculators NOT allowed.
This test contains **THREE** problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.

Problem 1. In the system below, the two analysis filters, $h_0[n]$ and $h_1[n]$, and the two synthesis filters, $f_0[n]$ and $f_1[n]$, form a Quadrature Mirror Filter (QMF). Specifically,

$$h_1[n] = (-1)^n h_0[n] \qquad f_0[n] = h_0[n] \qquad f_1[n] = -h_1[n]$$

The DTFT of the halfband filter $h_0[n]$ above may be expressed as follows:

$$H_0(\omega) = \begin{cases} 0, & -\pi < \omega < -\frac{3\pi}{4} \\ e^{j\frac{\omega}{2}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{3\pi}{4}+\omega}, & -\frac{3\pi}{4} < \omega < -\frac{\pi}{4} \\ e^{j\frac{\omega}{2}}, & |\omega| < \frac{\pi}{4} \\ e^{j\frac{\omega}{2}}\sqrt{\frac{2}{\pi}}\sqrt{\frac{3\pi}{4}-\omega}, & \frac{\pi}{4} < \omega < \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < \omega < \pi \end{cases}$$



Determine mathematically (include as much detail as possible) if the lowpass half-band filter above satisfies the condition required for Perfect Reconstruction. Be sure to clearly state what that condition is (don't need to rederive it) and then show whether it is satisfied with the filter $h_0[n]$, showing as much detail as possible.

Just need to assess if

(1) $H_0^2(\omega) - H_0^2(\omega - \pi) = c e^{-j n_0 \omega}$ for all ω

$$\boxed{H_0^2(\omega) = e^{j\omega}\frac{2}{\pi}\left(\frac{3\pi}{4} - \omega\right) \qquad \frac{\pi}{4} < \omega < \frac{3\pi}{4}}$$

- Just need to prove (1) for $0 < \omega < \pi$ since filter impulse response is real-valued
- Hence, $H_0(-\omega) = H_0^*(\omega)$
- Three regions to consider:
  1. $0 < \omega < \frac{\pi}{4}$
  2. $\frac{\pi}{4} < \omega < \frac{3\pi}{4}$
  3. $\frac{3\pi}{4} < \omega < \pi$

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See next page for plots

1. $0 < \omega < \frac{\pi}{4}$:

$$H_0^2(\omega) - H_0^2(\omega-\pi) = \left(e^{j\frac{\omega}{2}}\right)^2 - 0 = e^{j\omega} \checkmark$$

2. $\frac{\pi}{4} < \omega < \frac{3\pi}{4}$: $H_0^2(\omega) - H_0^2(\omega-\pi)$

$$= e^{j\omega}\frac{2}{\pi}\left(\frac{3\pi}{4}-\omega\right) - \left(e^{j\frac{(\omega-\pi)}{2}}\right)^2 \underbrace{\frac{2}{\pi}\left(\omega-\frac{\pi}{4}\right)}_{\text{see next page}}$$

$$= e^{j\omega}\frac{2}{\pi}\left\{\frac{3\pi}{4} - \omega - (-j)^2\left(\omega - \frac{\pi}{4}\right)\right\}$$

$$= e^{j\omega}\frac{2}{\pi}\left\{\frac{3\pi}{4} - \omega + \omega - \frac{\pi}{4}\right\} = e^{j\omega} \checkmark$$

3. $\frac{3\pi}{4} < \omega < \pi$: $H_0^2(\omega) - H_0^2(\omega-\pi)$

$$0 - \left(e^{j\frac{(\omega-\pi)}{2}}\right)^2$$

see next page

$$= -(-j)^2 e^{j\omega}$$

$$= e^{j\omega} \checkmark$$

all checks

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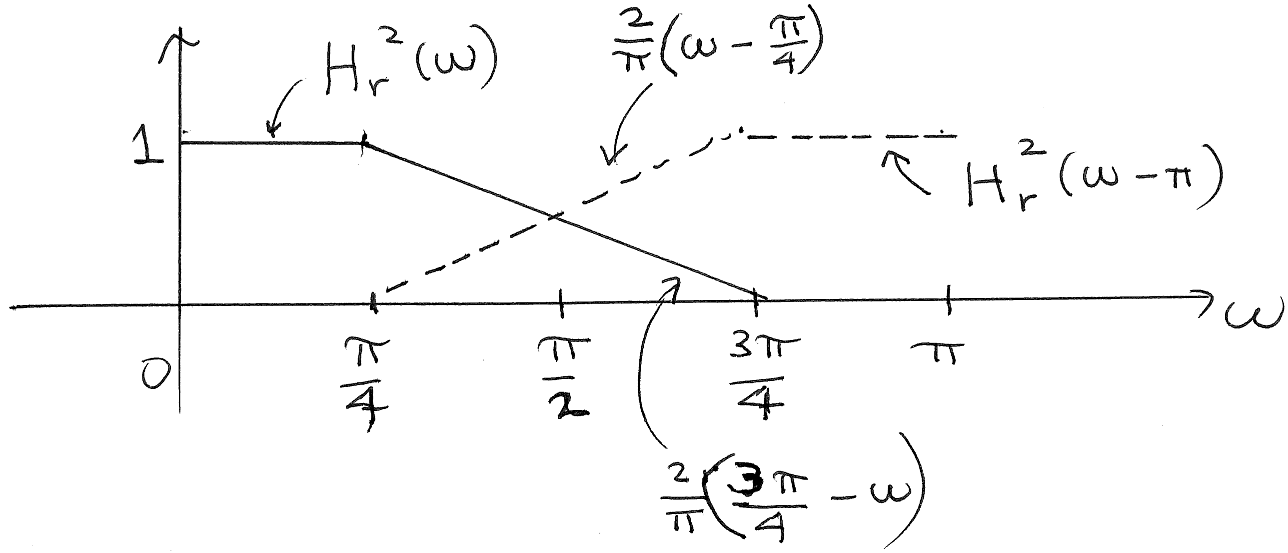
For plotting purposes, we will ignore $e^{j\frac{\omega}{2}}$ factor

$$H_o^2(\omega) = H_r^2(\omega)\, e^{j\omega}$$

plot $H_r^2(\omega)$ and $H_r^2(\omega - \pi)$ over $0 < \omega < \pi$



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Problem 2. For all parts of this problem, $x[n]$ is the fine-length sinewave of length $L = 8$ with frequency $\omega_o = \pi$ defined below, and $h[n]$ is a causal filter of length $M = 4$ which may be expressed in sequence form as $h[n] = \{1, -1, 1, -1\}$.

$$x[n] = e^{j\pi n}\{u[n] - u[n-8]\} \qquad h[n] = (-1)^n \{u[n] - u[n-4]\}$$

(a) Compute the linear convolution of $x[n]$ and $h[n]$. Indicate which points are the transient points (partial overlap) at the beginning and end, and also which points are "pure" sinewave (full overlap.)

(b) With $X_N(k)$ computed as the 8-pt DFT of $x[n]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 8$ values of the 8-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(c) Using your answer to (a), explain your answer to (b) by mathematically illustrating the time-domain aliasing effect.

(d) The product sequence $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 8$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$, computed according to Eqn (1) below:

$$Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)} \qquad (1)$$

(a) Since filter is of length 4, 3 transients points at beginning and end: In the full overlap region, we will have:

$$H(\pi)e^{j\pi n} \quad \text{for } n = 4, \ldots, 7$$

where: $$H(\omega) = \frac{\sin\left(\frac{4}{2}(\omega - \pi)\right)}{\sin\left(\frac{1}{2}(\omega - \pi)\right)} e^{-j\frac{(4-1)}{2}(\omega - \pi)}$$

which at $\omega = \pi$ is $H(\pi) = 4$

So, for $n = 4, \ldots, 7$ we have $4e^{j\pi n}$

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linear convolution:

$$y[n] = \{ \underbrace{1, -2, 3}_{\text{partial}}, \underbrace{-4, 4, -4, 4, -4}_{\text{full}}, \underbrace{3, -2, 1}_{\text{partial}} \}$$

(b) "Lucky" case for $x[n]$: $\pi = k \frac{2\pi}{8} \Rightarrow k = 4$

$$X_8[k] = 8\,\delta[k-4]$$

and $H_8(4) = H(\pi) = 4 \Rightarrow$ already computed

Thus: $y_8[n] = 4\, e^{j\pi n} \{ u[n] - u[n-8] \}$

(c) Last 3 points of $y[n]$ above aliased into first 3 points:

$$\boxed{\begin{matrix} y_8[n] = y[n] + y[n+8] \\ n = 0, 1, \ldots, 7 \end{matrix}}$$

$$\begin{array}{r} 1, -2, 3 \\ 3, -2, 1 \\ \hline 4, -4, 4 \end{array}$$

Thus:

$$y_8[n] = \{ \underset{\uparrow}{4}, -4, 4, -4, 4, -4, 4, -4 \}$$

(d) $Y_8(k)$ only has one nonzero value at $k=4$

Thus: $$Y_r(\omega) = 4 \frac{\sin\left(\frac{8}{2}(\omega - \pi)\right)}{\sin\left(\frac{1}{2}(\omega - \pi)\right)} e^{-j\frac{(7-1)}{2}(\omega - \pi)}$$

Problem 3. Consider a causal FIR filter of length $M = 12$ with impulse response as defined below:

$$h[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin\left[\frac{\pi}{4}(n+\ell 12)\right]}{\pi(n+\ell 12)} \frac{\sin\left[\frac{3\pi}{4}(n+\ell 12)\right]}{\pi(n+\ell 12)} \{u[n] - u[n-12]\}$$

(a) Determine all 12 numerical values of the 12-pt DFT of $h[n]$, denoted $H_{12}(k)$, for $0 \leq k \leq 11$. List the values clearly: $H_{12}(k) =?$, for $0 \leq k \leq 11$.

(b) Consider the sequence $x[n]$ of length $L = \boxed{12}$elow, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{11} e^{jk\frac{2\pi}{12}n} \{u[n] - u[n-12]\}$$

$y_{12}[n]$ is formed by computing $X_{12}(k)$ as an 12-pt DFT of $x[n]$, $H_{12}(k)$ as an 12-pt DFT of $h[n]$, and then $y_{12}[n]$ as the 12-pt inverse DFT of $Y_{12}(k) = X_{12}(k)H_{12}(k)$. Express the result $y_{12}[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

(c) Next, consider a causal signal of length $M = 12$ with impulse response as defined below:

$$x[n] = \sum_{\ell=-\infty}^{\infty} 8 \left\{ \frac{\sin\left[\frac{\pi}{4}(n+\ell 12)\right]}{\pi(n+\ell 12)} \right\}^2 \cos\left(\frac{\pi}{2}(n+\ell 12)\right) \{u[n] - u[n-12]\}$$

Determine all 12 numerical values of the 12-pt DFT of $x[n]$, denoted $X_{12}(k)$, for $0 \leq k \leq 11$. List all 12 numerical values clearly.

(d) **EXTRA CREDIT.** Next, consider a causal signal of length $M = 16$ with impulse response as defined below:

$$x[n] = \sum_{\ell=-\infty}^{\infty} 16j \left\{ \frac{\sin\left[\frac{\pi}{8}(n+\ell 16)\right]}{\pi(n+\ell 16)} \right\} \left\{ \frac{\sin\left[\frac{3\pi}{8}(n+\ell 16)\right]}{\pi(n+\ell 16)} \right\} \sin\left(\frac{\pi}{2}(n+\ell 16)\right) \{u[n] - u[n-16]\}$$

Determine all 16 numerical values of the 16-pt DFT of $x[n]$, denoted $X_{16}(k)$, for $0 \leq k \leq 15$. List all 16 numerical values clearly.

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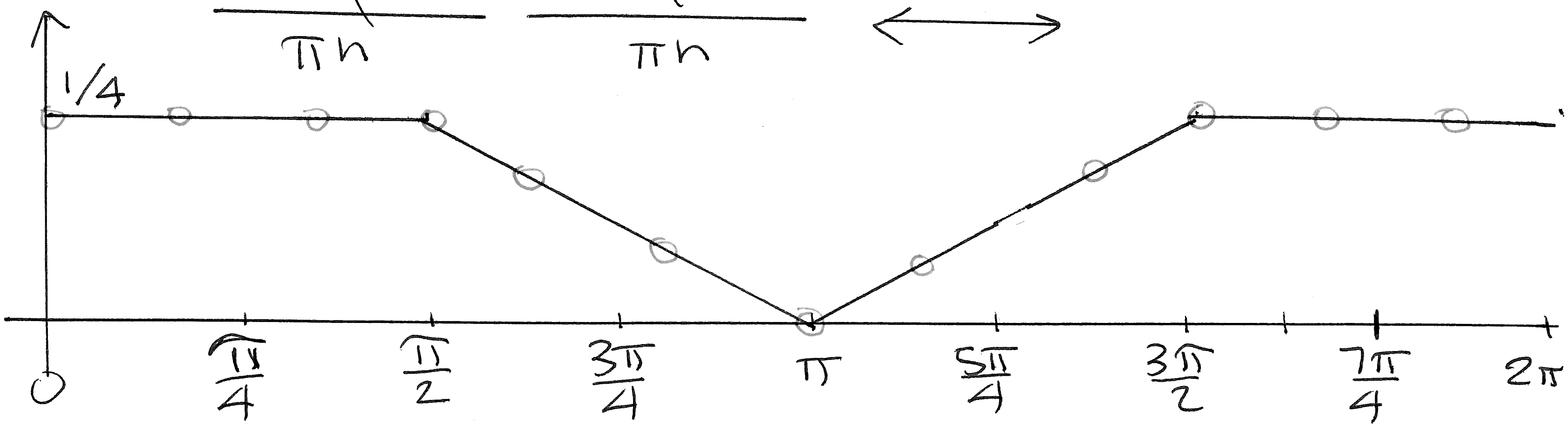
3 (a) $\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \quad \dfrac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n} \quad \overset{DTFT}{\longleftrightarrow}$



height $= \frac{1}{4}$ I meant to multiply by 4, but oh well :)

sample at $\omega_k = k\frac{2\pi}{12} = k\frac{\pi}{6}$ $k = 0, 1, \dots, 11$

$H_{12}(0) = H(0) = \frac{1}{4}$

$H_{12}(1) = H\left(\frac{\pi}{6}\right) = \frac{1}{4}$

$H_{12}(2) = H\left(\frac{\pi}{3}\right) = \frac{1}{4}$

$H_{12}(3) = H\left(\frac{\pi}{2}\right) = \frac{1}{4}$

$H_{12}(4) = H\left(\frac{2}{3}\pi\right) = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$

$H_{12}(5) = H\left(\frac{5\pi}{6}\right) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$

$H_{12}(6) = H(\pi) = 0$

$H_{12}(7) = H\left(\frac{7\pi}{6}\right) = \frac{1}{12}$

$H_{12}(8) = H\left(\frac{4\pi}{3}\right) = \frac{1}{6}$

$H_{12}(9) = H_{12}(10) = H_{12}(11) = \frac{1}{4}$

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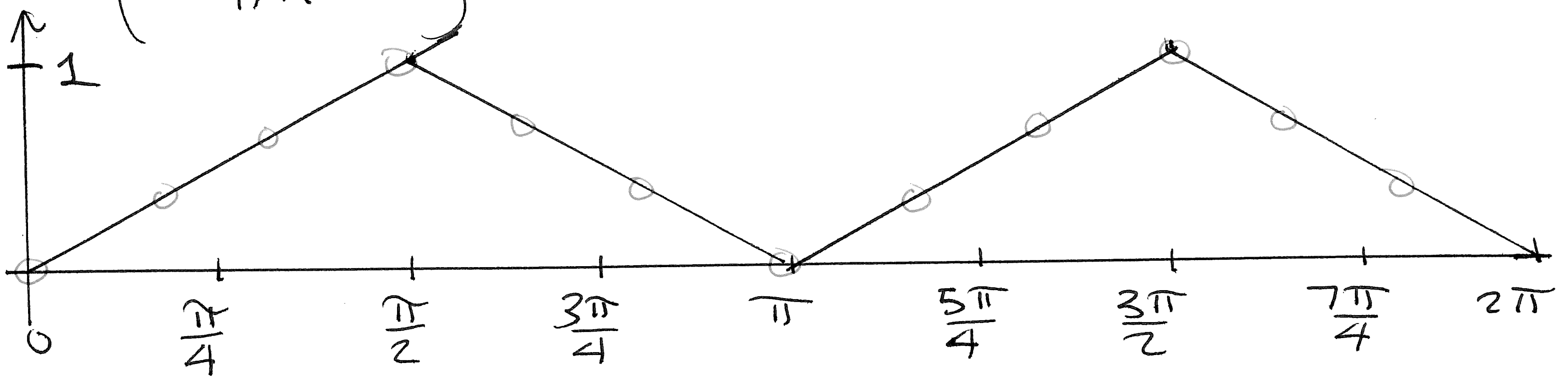
3 (b)

$$y_{12}[n] = \sum_{k=0}^{11} \underbrace{H_{12}(k)}_{\text{listed in part (a)}} e^{jk\frac{2\pi}{12}n} \{u[n] - u[n-12]\}$$

3 (c) From Exam 2, we have

$$8\left\{\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right\}^2 \cos\left(\frac{\pi}{2}n\right) \overset{DTFT}{\longleftrightarrow}$$



sampled at $\omega_k = k\frac{2\pi}{12} = k\frac{\pi}{6}$ $k = 0, 1, \ldots, 11$

$X_{12}(0) = 0$

$X_{12}(1) = 1/3$

$X_{12}(2) = 2/3$

$X_{12}(3) = 1$

$X_{12}(4) = 2/3$

$X_{12}(5) = 1/3$

$X_{12}(6) = 0$

$X_{12}(7) = 1/3$

$X_{12}(8) = 2/3$

$X_{12}(9) = 1$

$X_{12}(10) = 2/3$

$X_{12}(11) = 1/3$

3 (d)

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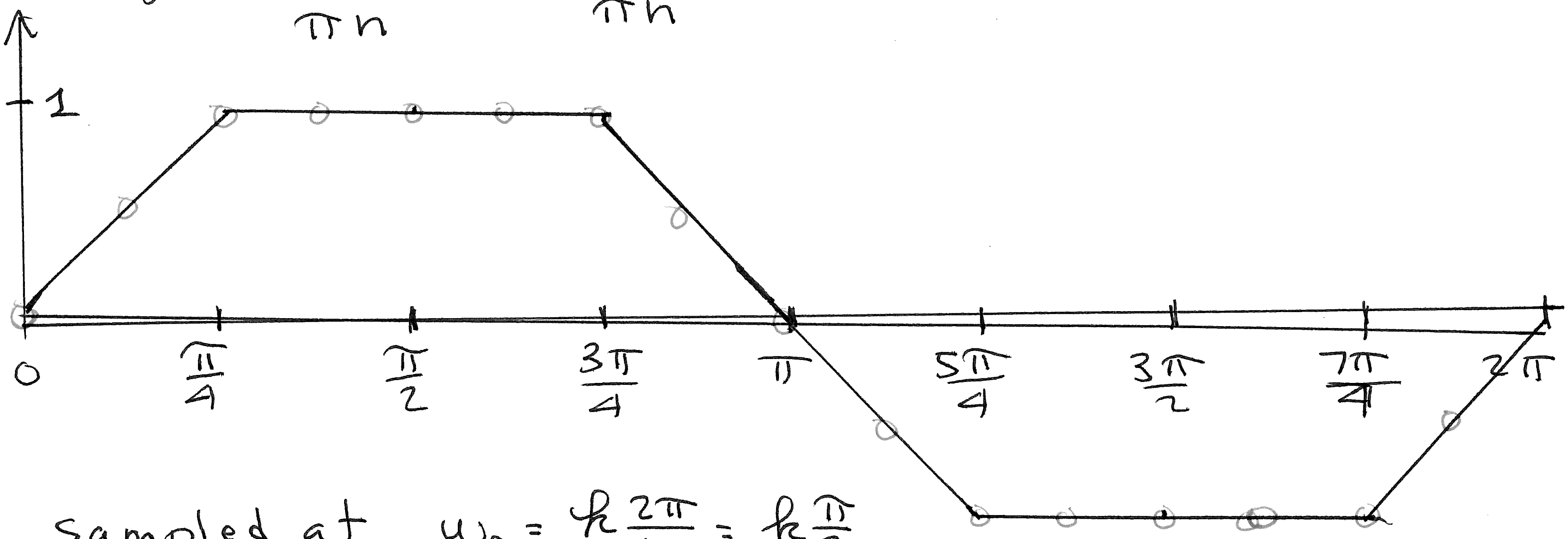
From Exam 2, we know that

$$16j \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n} \frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \sin\left(\frac{\pi}{2}n\right) \overset{DTFT}{\longleftrightarrow}$$



sampled at $\omega_k = k\frac{2\pi}{16} = k\frac{\pi}{8}$

$k = 0, 1, \ldots, 15$

$X_{16}(0) = 0$ $\qquad X_{16}(8) = 0$

$X_{16}(1) = .5$ $\qquad X_{16}(9) = -.5$

$X_{16}(2) = 1$ $\qquad X_{16}(10) = -1$

$X_{16}(3) = 1$ $\qquad X_{16}(11) = -1$

$X_{16}(4) = 1$ $\qquad X_{16}(12) = -1$

$X_{16}(5) = 1$ $\qquad X_{16}(13) = -1$

$X_{16}(6) = 1$ $\qquad X_{16}(14) = -1$

$X_{16}(7) = .5$ $\qquad X_{16}(15) = -.5$