SOLUTION

ANSWER KEY

NAME: 30 Nov. 2011

Digital Signal Processing I — Exam 3 — Fall 2011

Session 40 — 30 Nov. 2011

Cover Sheet

WRITE YOUR NAME ON EACH EXAM SHEET

Test Duration: 60 minutes.

Open Book but Closed Notes.

Calculators NOT allowed.

This test contains **two** problems.

All work should be done in the space provided.

Do **not** just write answers; provide concise reasoning for each answer.

Problem 1. Let $x[n]$ be a discrete-time rectangular pulse of length $L = 5$ and $h[n]$ be a discrete-time rectangular pulse of length $M = 3$ as defined below:

$$x[n] = u[n] - u[n-5] \qquad\qquad h[n] = u[n] - u[n-3]$$

(a) With $X_N(k)$ computed as the 5-pt DFT of $x[n] = u[n] - u[n-5]$ and $H_N(k)$ computed as the 5-pt DFT of $h[n] = u[n] - u[n-3]$. The 5-point sequence $y_5[n]$ is computed as the 5-pt inverse DFT of the product $Y_N(k) = X_N(k)H_N(k)$. Write out the 5 numerical values of $y_5[n]$ in sequence form as $\{y_5[0], y_5[1], y_5[2], y_5[3], y_5[4]\}$.

$$y_5[n] = x[n] \; ⑤ \; \underbrace{h[n]}_{\text{pad 2 zeroes}}$$

Use circulant matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

<u>or</u> use time-domain aliasing formula:

$$y[n] = x[n] * h[n] = \{1, 2, 3, 3, 3, 2, 1\}$$

$$y_5[n] = y[n] + y[n+5] \Rightarrow \text{last two entries aliased into first two entries}$$

$$y_5[n] = \{3, 3, 3, 3, 3\}$$

(b) With $X_N(k)$ computed as the 8-pt DFT of $x[n] = u[n] - u[n-5]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n] = u[n] - u[n-3]$. The 8-point sequence $y_8[n]$ is computed as the 8-pt inverse DFT of the product $Y_N(k) = X_N(k)H_N(k)$. Write out the 8 numerical values of $y_8[n]$ in sequence form.

linear convolution of length $5+3-1=7$

since $N = 8 > 7 \Rightarrow$ no time-domain aliasing

$y_8[n] = \{1, 2, 3, 3, 3, 2, 1, 0\}$

(c) With $X_N(k)$ computed as the 10-pt DFT of $x[n] = u[n] - u[n-5]$ and $H_N(k)$ computed as the 10-pt DFT of $h[n] = u[n] - u[n-3]$. The 10-point sequence $y_{10}[n]$ is computed as the 10-pt inverse DFT of the product $Y_N(k) = X_N(k)H_N(k)$. Write out the 10 numerical values of $y_{10}[n]$ in sequence form.

$N = 10 > 7 \Rightarrow$

$y_{10}[n] = \{1, 2, 3, 3, 3, 2, 1, 0, 0, 0\}$

Problem 2.

For all parts of this problem, the reconstructed spectrum is computed according to the equation below:

$$Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)} \tag{1}$$

Let $x[n]$ be a finite-length sinewave of length $L = 8$ and $h[n]$ be a discrete-time rectangular pulse of length $M = 5$ as defined below:

$$x[n] = e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\} \qquad\qquad h[n] = u[n] - u[n-5]$$

(a) With $X_N(k)$ computed as the 16-pt DFT of $x[n]$ and $H_N(k)$ computed as the 16-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 16$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

linear convolution is of length $8+5-1 = 12$

since $N = 16 > 12 \Rightarrow$ no time-domain aliasing

$\Rightarrow$ perfect reconstruction

Thus, $Y_r(\omega) = Y(\omega) = X(\omega)\,H(\omega)$

$$Y(\omega) = \underbrace{\frac{\sin\left(\frac{8}{2}\left(\omega - \frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{\pi}{2}\right)\right)} e^{-j\frac{(8-1)}{2}\left(\omega - \frac{\pi}{2}\right)}}_{X(\omega)} \; \underbrace{\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(5-1)}{2}\omega}}_{H(\omega)}$$

(b) With $X_N(k)$ computed as the 12-pt DFT of $x[n] = e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\}$ and $H_N(k)$ computed as the 12-pt DFT of $h[n] = u[n]-u[n-5]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 12$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

Again, $N = 12 = 12$ length of linear convolution

$Y_r(\omega) = Y(\omega)$ same answer as 2 (a)

(c) The answer to this part will be useful in determining the answer to part (d). $X_N(k)$ computed as the 8-pt DFT of $x[n] = e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\}$ and $H_N(k)$ computed as the 8-pt DFT of $h[n] = u[n] - u[n-5]$. Develop and delineate your answers to each of the four steps below in the space below. Simplify each answer as much as possible.

(i) Determine a closed-form expression for the 8-pt DFT, $X_N(k)$, of $x[n] = e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\}$.

(ii) Determine a closed-form expression for the 8-pt DFT, $H_N(k)$, of $h[n] = \{u[n] - u[n-5]\}$.

(iii) Determine a closed-form expression for the product $Y_N(k) = X_N(k)H_N(k)$.

(iv) Determine a simple, closed-form expression for $y_8[n]$ equal to the 8-pt inverse DFT of $Y_N(k) = X_N(k)H_N(k)$. Note that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

(i) $\frac{\pi}{2} = k\frac{2\pi}{8} \Rightarrow k = 2$

special case: $X_N(k) = 8\,\delta[k-2]$

(ii) $H_N(k) = H(\omega)\Big|_{\omega = k\frac{2\pi}{8}}$

$$= \frac{\sin\left(\frac{5}{2}k\frac{2\pi}{8}\right)}{\sin\left(\frac{1}{2}k\frac{2\pi}{8}\right)} e^{-j\frac{(5-1)}{2}k\frac{2\pi}{8}} = \frac{\sin\left(k\frac{5\pi}{8}\right)}{\sin\left(k\frac{\pi}{8}\right)} e^{-jk\frac{\pi}{2}}$$

(iii) $Y_N(k) = 8\,\delta[k-2]\,H_N(k)$

$$= 8\,H_N(2)\,\delta[k-2]$$

$$= 8\,\frac{\sin\left(\frac{10\pi}{8}\right)}{\sin\left(\frac{2\pi}{8}\right)} e^{-j\frac{2\pi}{2}}\,\delta[k-2]$$

$$= 8\,\frac{-1/\sqrt{2}}{1/\sqrt{2}}\,e^{-j\pi}\,\delta[k-2] = +8\,\delta[k-2]$$

(iv)

$$y_8[n] = H_N(2)\,e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\}$$

$$= e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\}$$

(d) With $X_N(k)$ computed as the 8-pt DFT of $x[n] = e^{j\frac{\pi}{2}n}\{u[n] - u[n-8]\}$ and $H_N(k)$ computed as the 8-pt DFT of $h[n] = u[n] - u[n-5]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 8$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$. Note that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Since $Y_N(k) = 8 \quad \delta[k-2]$

There's only one nonzero term in the spectral reconstruction, the $k=2$ term

$$Y_r(\omega) = Y_N(2) \frac{\sin\left[\frac{8}{2}\left(\omega - \frac{2\pi 2}{8}\right)\right]}{8\sin\left[\frac{1}{2}\left(\omega - \frac{2\pi 2}{8}\right)\right]} e^{-j\frac{(8-1)}{2}\left(\omega - \frac{\pi}{2}\right)}$$

$$= \cancel{8} \cdot \frac{\sin\left[4\left(\omega - \frac{\pi}{2}\right)\right]}{\cancel{8}\sin\left[\frac{1}{2}\left(\omega - \frac{\pi}{2}\right)\right]} e^{-j\frac{7}{2}\left(\omega - \frac{\pi}{2}\right)}$$

(The 8's cancel)
and (-1)'s cancel

$$Y_r(\omega) = \frac{\sin\left(4\left(\omega - \frac{\pi}{2}\right)\right]}{\sin\left(\frac{1}{2}\left(\omega - \frac{\pi}{2}\right)\right]} e^{-j\frac{7}{2}\left(\omega - \frac{\pi}{2}\right)}$$