NAME: Digital Signal Processing I Exam 3 Fall 2011
Session 40 30 Nov. 2011
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Cover Sheet

WRITE YOUR NAME ON EACH EXAM SHEET

Test Duration: 60 minutes.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains two problems.
All work should be done in the space provided.
Do not just write answers; provide concise reasoning for each answer.
Problem 1. Let \( x[n] \) be a discrete-time rectangular pulse of length \( L = 5 \) and \( h[n] \) be a discrete-time rectangular pulse of length \( M = 3 \) as defined below:

\[
x[n] = u[n] - u[n - 5] \\
h[n] = u[n] - u[n - 3]
\]

(a) With \( X_N(k) \) computed as the 5-pt DFT of \( x[n] = u[n] - u[n - 5] \) and \( H_N(k) \) computed as the 5-pt DFT of \( h[n] = u[n] - u[n - 3] \). The 5-point sequence \( y_5[n] \) is computed as the 5-pt inverse DFT of the product \( Y_N(k) = X_N(k)H_N(k) \). Write out the 5 numerical values of \( y_5[n] \) in sequence form as \( \{y_5[0], y_5[1], y_5[2], y_5[3], y_5[4]\} \).

\[
y_5[n] = x[n] \overset{(5)}{\underset{\text{pad 2 zeroes}}{\oplus}} h[n]
\]

Use circulant matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
3 \\
3 \\
3 \\
3 \\
3
\end{bmatrix}
\]

or use time-domain aliasing formula:

\[
y_5[n] = x[n] \ast h[n] = \{1, 3, 3, 3, 3, 3, 3, 3\}
\]

\[
y_5[5] = y_5[n] + y_5[n + 5] \Rightarrow \text{last two entries aliased into first two entries}
\]

\[
y_5' = \{3, 3, 3, 3, 3\}
\]
(b) With $X_N(k)$ computed as the 8-pt DFT of $x[n] = u[n] - u[n - 5]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n] = u[n] - u[n - 3]$. The 8-point sequence $y_8[n]$ is computed as the 8-pt inverse DFT of the product $Y_N(k) = X_N(k)H_N(k)$. Write out the 8 numerical values of $y_8[n]$ in sequence form.

linear convolution of length $5 + 3 - 1 = 7$

since $N = 8 > 7 \Rightarrow$ no time-domain aliasing

$y_8[n] = \{1, 2, 3, 3, 3, 2, 1, 0\}$
(c) With $X_N(k)$ computed as the 10-pt DFT of $x[n] = u[n] - u[n-5]$ and $H_N(k)$ computed as the 10-pt DFT of $h[n] = u[n] - u[n-3]$. The 10-point sequence $y_{10}[n]$ is computed as the 10-pt inverse DFT of the product $Y_N(k) = X_N(k)H_N(k)$. Write out the 10 numerical values of $y_{10}[n]$ in sequence form.

$$N = 10 > 7 \implies y_{10}[n] = \{1, 2, 3, 3, 3, 2, 1, 0, 0, 0\}$$
Problem 2.

For all parts of this problem, the reconstructed spectrum is computed according to the equation below:

\[ Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin \left[ \frac{N}{2} \left( \omega - \frac{2\pi k}{N} \right) \right]}{N \sin \left[ \frac{1}{2} \left( \omega - \frac{2\pi k}{N} \right) \right]} e^{-j \frac{N-1}{2} \left( \omega - \frac{2\pi k}{N} \right)} \]  \hspace{1cm} (1)

Let \( x[n] \) be a finite-length sinewave of length \( L = 8 \) and \( h[n] \) be a discrete-time rectangular pulse of length \( M = 5 \) as defined below:

\[ x[n] = e^{j \frac{\pi}{4} n} \{ u[n] - u[n - 8] \} \quad \text{and} \quad h[n] = u[n] - u[n - 5] \]

(a) With \( X_N(k) \) computed as the 16-pt DFT of \( x[n] \) and \( H_N(k) \) computed as the 16-pt DFT of \( h[n] \), the product \( Y_N(k) = X_N(k)H_N(k) \) is used in Eqn (1) with \( N = 16 \). Write a closed-form expression for the reconstructed spectrum \( Y_r(\omega) \).

Linear convolution is of length \( B + S - 1 = 12 \) since \( N = 16 > 12 \Rightarrow \) no time-domain aliasing \( \Rightarrow \) perfect reconstruction

Thus,

\[ Y_r(\omega) = Y(\omega) = X(\omega)H(\omega) \]

\[ Y(\omega) = \frac{\sin \left( \frac{B}{2} \left( \omega - \frac{\pi}{2} \right) \right)}{\sin \left( \frac{1}{2} \left( \omega - \frac{\pi}{2} \right) \right)} e^{-j \frac{B-1}{2} \left( \omega - \frac{\pi}{2} \right)} \]

\[ \frac{\sin \left( \frac{\pi}{2} \omega \right)}{\sin \left( \frac{1}{2} \omega \right)} \]

\[ \frac{\sin \left( \frac{\pi}{2} \omega \right)}{\sin \left( \frac{1}{2} \omega \right)} \]

\[ X(\omega) \quad H(\omega) \]
(b) With $X_N(k)$ computed as the 12-pt DFT of $x[n] = e^{j\frac{\pi}{8} n \{u[n] - u[n - 8]\}}$ and $H_N(k)$ computed as the 12-pt DFT of $h[n] = u[n] - u[n - 5]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 12$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

Again, $N = 12 = 12$ length of linear convolution

$Y_r(\omega) = Y(\omega)$ same answer as 2 (a)
(c) The answer to this part will be useful in determining the answer to part (d). $X_N(k)$ computed as the 8-pt DFT of $x[n] = e^{j\frac{2\pi}{8}} \{u[n] - u[n - 8]\}$ and $H_N(k)$ computed as the 8-pt DFT of $h[n] = u[n] - u[n - 5]$. Develop and delineate your answers to each of the four steps below in the space below. Simplify each answer as much as possible.

(i) Determine a closed-form expression for the 8-pt DFT, $X_N(k)$, of $x[n] = e^{j\frac{2\pi}{8}} \{u[n] - u[n - 8]\}$.
(ii) Determine a closed-form expression for the 8-pt DFT, $H_N(k)$, of $h[n] = \{u[n] - u[n - 5]\}$.
(iii) Determine a closed-form expression for the product $Y_N(k) = X_N(k)H_N(k)$.
(iv) Determine a simple, closed-form expression for $y_8[n]$ equal to the 8-pt inverse DFT of $Y_N(k) = X_N(k)H_N(k)$. Note that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

\[
\frac{\pi}{2} = k\frac{2\pi}{8} \implies k = 2
\]

Special case: $X_N(k) = 8 \delta[k - 2]

\[
H_N(k) = H(\omega) \bigg|_{\omega = k\frac{2\pi}{8}}
\]

\[
= \frac{\sin\left(\frac{\pi}{2} - k\frac{2\pi}{8}\right)}{\sin\left(\frac{1}{2}k\frac{2\pi}{8}\right)} e^{-j \frac{(5\pi)}{2} k\frac{2\pi}{8}} = \frac{\sin\left(k\frac{5\pi}{8}\right)}{\sin\left(k\frac{\pi}{8}\right)} e^{-j \frac{\pi}{2}}
\]

\[
Y_N(k) = 8 \delta[k - 2] H_N(k)
\]

\[
= 8 \delta[k - 2] H_N(2) = 8 \delta[k - 2]
\]

\[
= 8 \frac{\sin\left(\frac{10\pi}{8}\right)}{\sin\left(\frac{2\pi}{8}\right)} e^{-j \frac{2\pi}{8}} \delta[k - 2] = 8 \frac{-1/\sqrt{2}}{1/\sqrt{2}} e^{-j \pi} \delta[k - 2] = 8 \delta[k - 2]
\]

\[
y_8(n) = H_N(2) e^{j\frac{\pi}{4}} \{u(n) - u(n - 8)\}
\]

\[
= e^{j\frac{\pi}{4}} \{u(n) - u(n - 8)\}
\]
(d) With $X_N(k)$ computed as the 8-pt DFT of $x[n] = e^{j\frac{\pi}{8} n} \{u[n] - u[n - 8]\}$ and $H_N(k)$ computed as the 8-pt DFT of $h[n] = u[n] - u[n - 5]$, the product $Y_N(k) = X_N(k)H_N(k)$ is used in Eqn (1) with $N = 8$. Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$. Note that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Since $Y_N(k) = 8 \delta(k-2)$

There's only one nonzero term in the spectral reconstruction, the $k=2$ term

$$Y_r(\omega) = Y_N(2) \frac{\sin\left[\frac{\omega}{2}(\omega - \frac{2\pi - 2}{8})\right]}{8 \sin\left[\frac{1}{2}(\omega - \frac{2\pi - 2}{8})\right]} e^{-j\frac{8}{2}(\omega - \frac{\pi}{2})}$$

$$= 8 \frac{\sin\left[4(\omega - \frac{\pi}{2})\right]}{8 \sin\left[\frac{1}{2}(\omega - \frac{\pi}{2})\right]} e^{-j\frac{\pi}{2}(\omega - \frac{\pi}{2})}$$

(The 8's cancel)

and (-1)'s cancel

$$Y_r(\omega) = \frac{\sin\left(4(\omega - \frac{\pi}{2})\right)}{\sin\left(\frac{1}{2}(\omega - \frac{\pi}{2})\right)} e^{-j\frac{\pi}{2}(\omega - \frac{\pi}{2})}$$