ECE538 Exam 3 Take-Home Fall 2025
Digital Signal Processing I Due: 7 December 2025

Cover Sheet

Test Duration: Due uploaded to Brightspace by 11:59 EST, Dec. 7 .
This test contains **three** problems.
You must show ALL work for each problem to receive full credit.

No.	**Topic(s) of Problem**	**Points**
1.	OFDM	25
2.	Noble's Identities	25
3.	Perfect Reconstruction Filter Bank	50

Problem 1. An OFDM signal is synthesized as a sum of eight sinewaves of length 8 as

$$\tilde{x}[n] = \frac{1}{8}\sum_{k=0}^{7} b_k e^{j2\pi\frac{k}{8}n}\{u[n] - u[n-8]\}$$

Each of the 8 values b_k, is one of the 8 values listed in the symbol alphabet below.

$$b_k \in \{-7, -5, -3, -1, 1, 3, 5, 7\} \quad k = 0, 1, 2, 3, 4, 5, 6, 7$$

We know in advance that the signal we transmit will be convolved with a filter of length $L = 3$. Thus, we add a cyclic prefix of length 3 which effectively creates a sum of sinewaves of length 11 as prescribed below (note the division by 8)

$$x[n] = \frac{1}{8}\sum_{k=0}^{7} b_k e^{j2\pi\frac{k}{8}n}\{u[n+3] - u[n-8]\}$$

The sequence of length 11 above is convolved with the filter below of length 3

$$h[n] = \{1, 2, 1\} = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

This ultimately yields the following sequence of length 11+3-1 = 13

$$y[n] = x[n] * h[n] =$$

$$\{0.9571, 0.6642, -2.0000, 1.5858, 6.5858, 1.5858, -2.0000, -0.5858, -0.5858, -0.5858, -2.0000, -2.1642, -0.4571\}$$

Your task is to determine the numerical values of each of the four symbols: b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7. You must use the OFDM methodology developed in class. Explain and show all the steps in determining your answers. Please lay your work out nicely with logical ordering. You can use Matlab or a calculator to do some of the number crunching, but it's important that you lay out all the steps. **NOTE:** you will not be able to determine one of the values in the normal OFDM way because one of the sinewave frequencies aligns with a null in the channel. Which value is it? And can you determine that value from the transient points at the two ends? Give it a try.

This page left intentionally blank for student work.

This page left intentionally blank for student work.

Problem 2. Convert each chain in the tree-structure Filter Bank in Figure 2(a) to achieve the regular structure in Figure 2(b) with the two complex-valued halfband filters, $h_0^{(2)}[n]$ and $h_1^{(2)}[n]$, defined below.

$$h_0^{(2)}[n] = e^{j\frac{\pi}{2}n}\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \qquad\qquad h_1^{(2)}[n] = e^{-j\frac{\pi}{2}n}\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

(A) 6 plots: plot (i) $H_0^{(2)}(\omega)$, (ii) $H_0^{(2)}(2\omega)$, (iii) $H_0^{(2)}(4\omega)$, (iv) $H_1^{(2)}(\omega)$, (v) $H_1^{(2)}(2\omega)$, (vi) $H_1^{(2)}(4\omega)$.

(B) Express each $H_k(\omega)$, $k = 0, 1, 2, 3, 4, 5, 6, 7$, in terms of (i) $H_0^{(2)}(\omega)$, (ii) $H_0^{(2)}(2\omega)$, (iii) $H_0^{(2)}(4\omega)$, (iv) $H_1^{(2)}(\omega)$, (v) $H_1^{(2)}(2\omega)$, and (vi) $H_1^{(2)}(4\omega)$.

(C) 8 plots: $H_0(\omega)$ $H_1(\omega)$, $H_2(\omega)$, $H_3(\omega)$, $H_4(\omega)$, $H_5(\omega)$, $H_6(\omega)$ and $H_7(\omega)$.
You do NOT have to draw a bunch of block diagrams. Use what you learned in Matlab Hmwk 3.

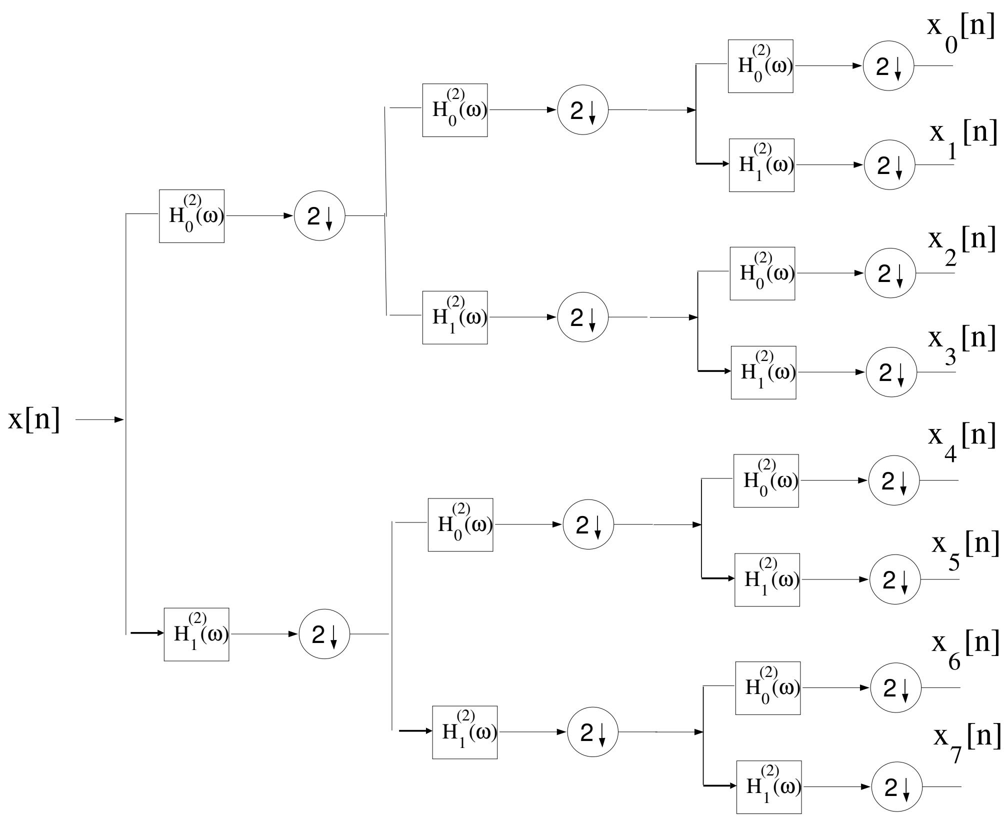

Figure 2(a). Analysis Section of Three-Stage Tree-Structured Filter Bank.

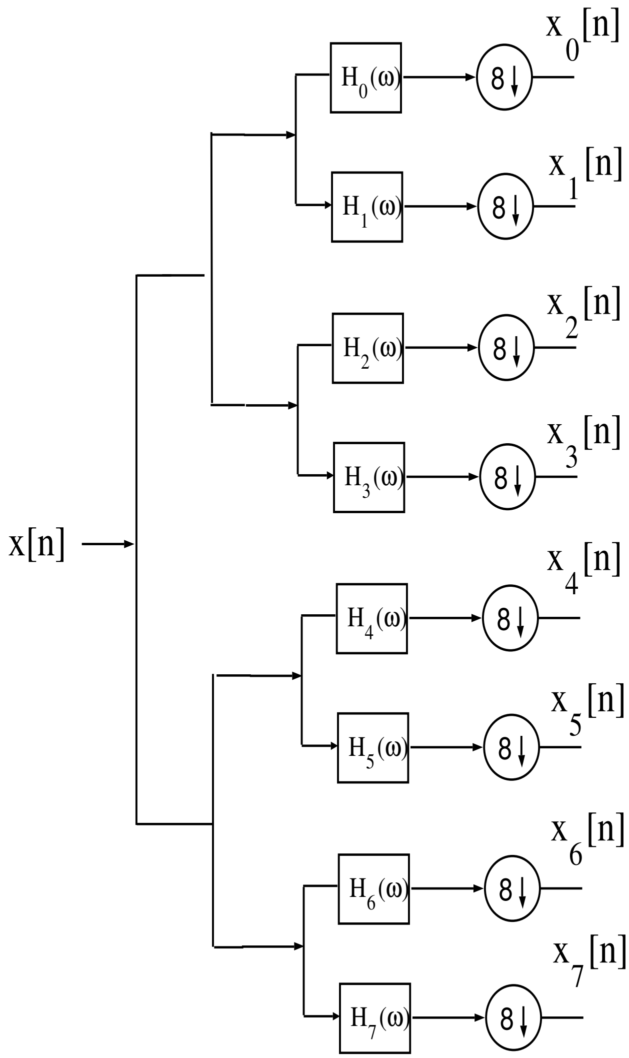

Figure 2(b). Analysis Filter Bank, $M = 8$.

This page left intentionally blank for student work for Problem 2.

This page left intentionally blank for student work for Problem 2.

This page left intentionally blank for student work.

Problem 3. Consider the $M = 4$ channel Filter Bank in Figure 3.1 on the next page. Review the derivation of the 2-channel QMF-based PRFB course notes posted at the course web site (https://engineering.purdue.edu/ ee538/TwoChannelQMF.pdf) You are to generalize that derivation for the case of 4 channels (Fig. 3.1) but for the specific case where the four real-valued filters are of length 4 as defined below.

(i) Using (https://engineering.purdue.edu/ ee538/VIP_MultirateFormulas.pdf) VIP Multirate Formulas, develop the matrix set of equations $\mathbf{V} = \mathbf{HX}$ where $\mathbf{V} = [V_0(\omega), V_1(\omega), V_2(\omega), V_3(\omega)]^T$ is 4x1, $\mathbf{H}$ is 4x4 and contains $H_k(\omega)$, k=0,1,2,3, and frequency shifted versions of $H_k(\omega)$ as entries, and $\mathbf{X}$ is 4x1 containing $X(\omega)$ and frequency shifted versions of $X(\omega)$ as entries. Show all work: this is your chance to show you know to apply all the Multirate Formulas derived in class.

(ii) Determine whether Perfect Reconstruction is achieved with the four **causal** length-4 analysis filters listed below and the corresponding four synthesis filters $g_k[n] = h_k[-n], k = 0, 1, 2, 3$. Show all work. **Note:** achieving Perfect Reconstruction means the output is the same as the input except for possibly an amplitude-scaling and an integer-delay (time-shift.)

$$h_0[n] = \{14, -2, -3, -4\}$$

$$h_1[n] = \{-2, 11, -6, -8\}$$

$$h_2[n] = \{-3, -6, 6, -12\}$$

$$h_3[n] = \{-4, -8, -12, -1\}$$

(iii) Determine and draw a computationally efficient implementation for the analysis side, using the four analysis filters defined in part (ii.)

(iv) Determine and draw a computationally efficient implementation for the synthesis side, four synthesis filters $g_k[n] = h_k[-n], k = 0, 1, 2, 3$, where $h_k[n]$ are the four analysis filters defined in part (ii.)

Note, the four analysis filters and the four synthesis filters are all real-valued and EACH is of length 4. Every quantity in this problem is real-valued. **SHOW ALL WORK: an answer with no supporting work will receive very little partial credit. This is a chance for you to show what you know about the efficient implementation of "filtering followed by decimation" AND the efficient implementation of "zero-inserts followed by filtering."**

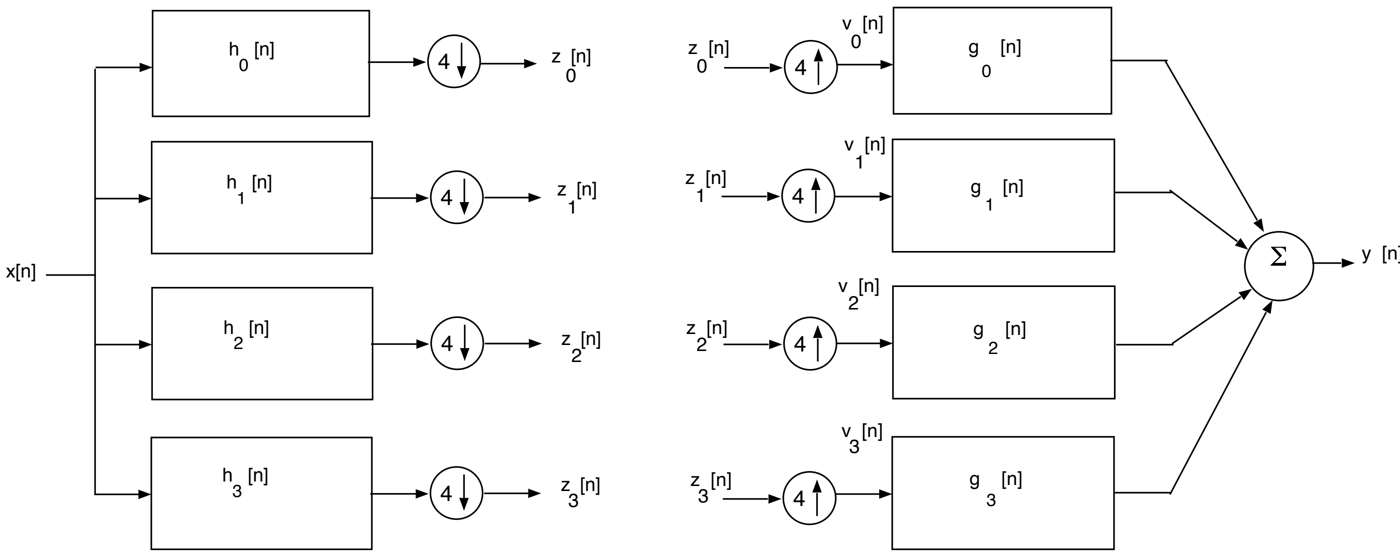

Figure 1(a). Analysis Filter Bank. Figure 1(b). Synthesis Filter Bank.

Figure 3.1. Problem 3.

This page left intentionally blank for student work for Problem 3.

This page left intentionally blank for student work for Problem 3.

This page left intentionally blank for student work for Problem 3.