Exam 3 Condensed Solution

Prob. 1

$$h[n] = \{\underset{\uparrow\, n=0}{8}, 4, 2\} \overset{\text{DFT}}{\underset{4}{\longleftrightarrow}} \{14, 6-4j, 6, 6+4j\} = H_4[k]$$

$\omega = 0 \quad \omega = \frac{\pi}{2} \quad \omega = \pi \quad \omega = \frac{3\pi}{2}$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -1 \end{bmatrix} \begin{bmatrix} 5-j \\ 7+3j \\ -1+5j \\ 3+7j \end{bmatrix} \right\} ./ \begin{bmatrix} 14 \\ 6-4j \\ 6 \\ 6+4j \end{bmatrix}$$

($./$: Matlab for pointwise division)

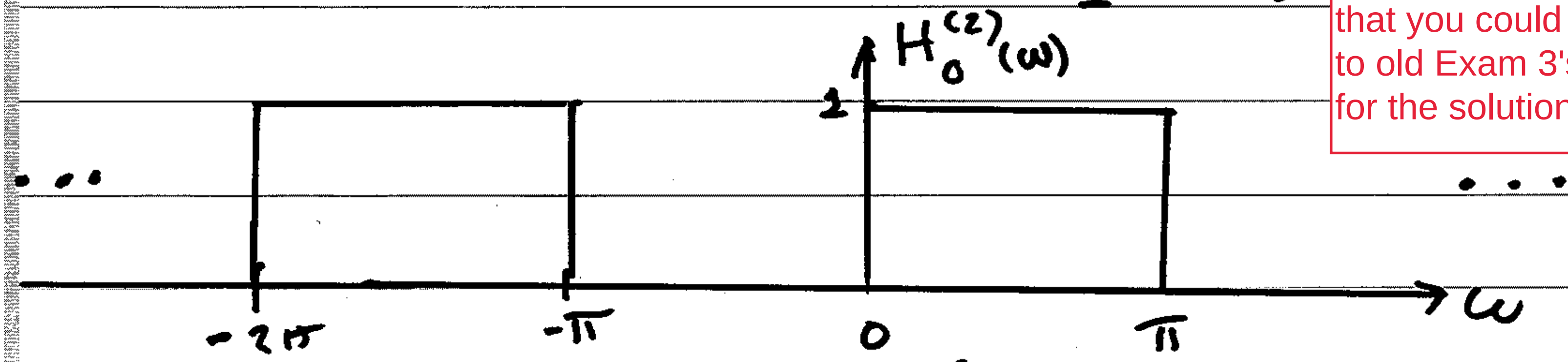
answers:

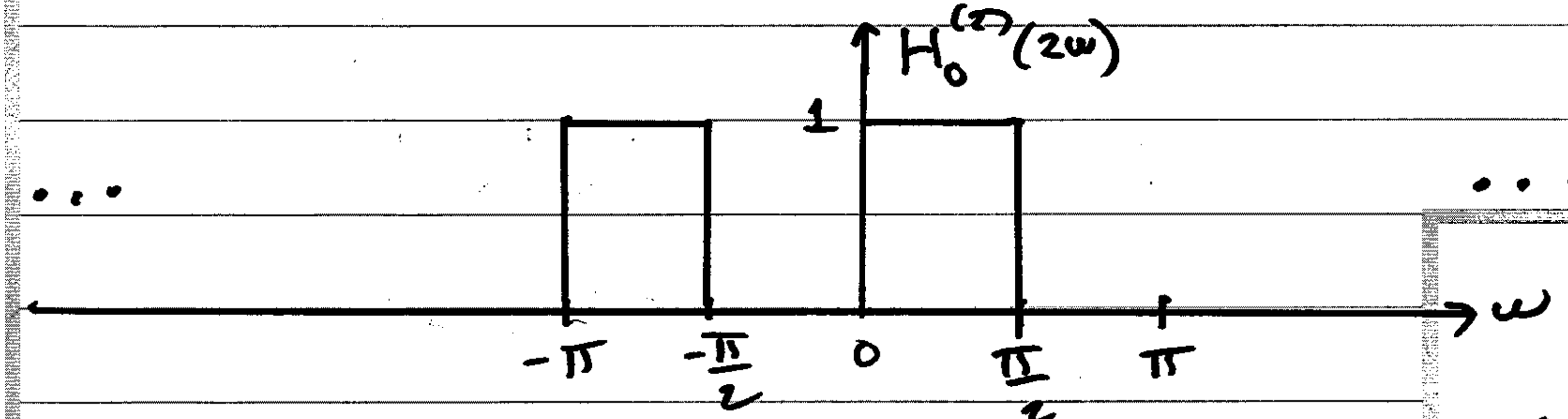
$b_0 = 1+j, \; b_1 = 1-j, \; b_2 = -1-j, \; b_3 = 1-j$

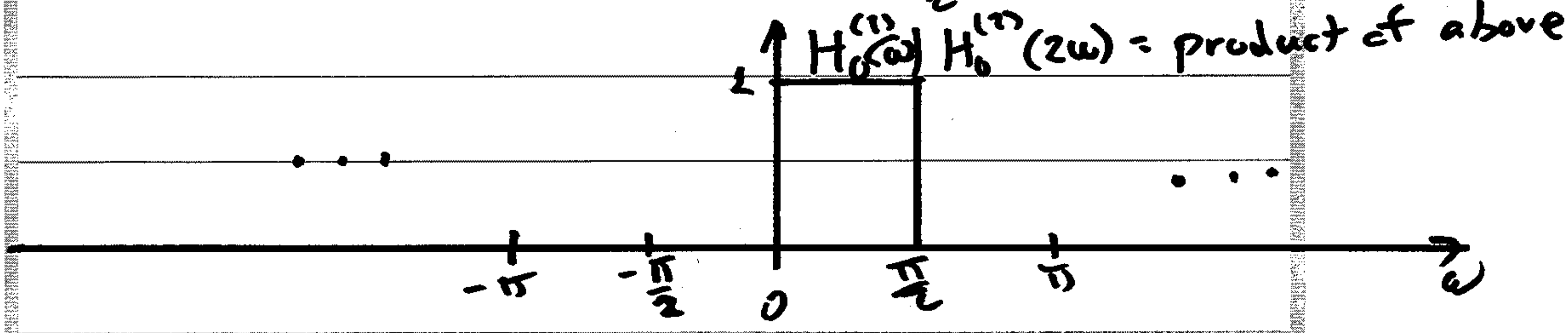
Prob. 2 Condensed Solution (Answer Key) (2a)

$$H_1(\omega) = H_0^{(2)}(\omega)\, H_0^{(2)}(2\omega)\, H_1^{(2)}(4\omega)$$

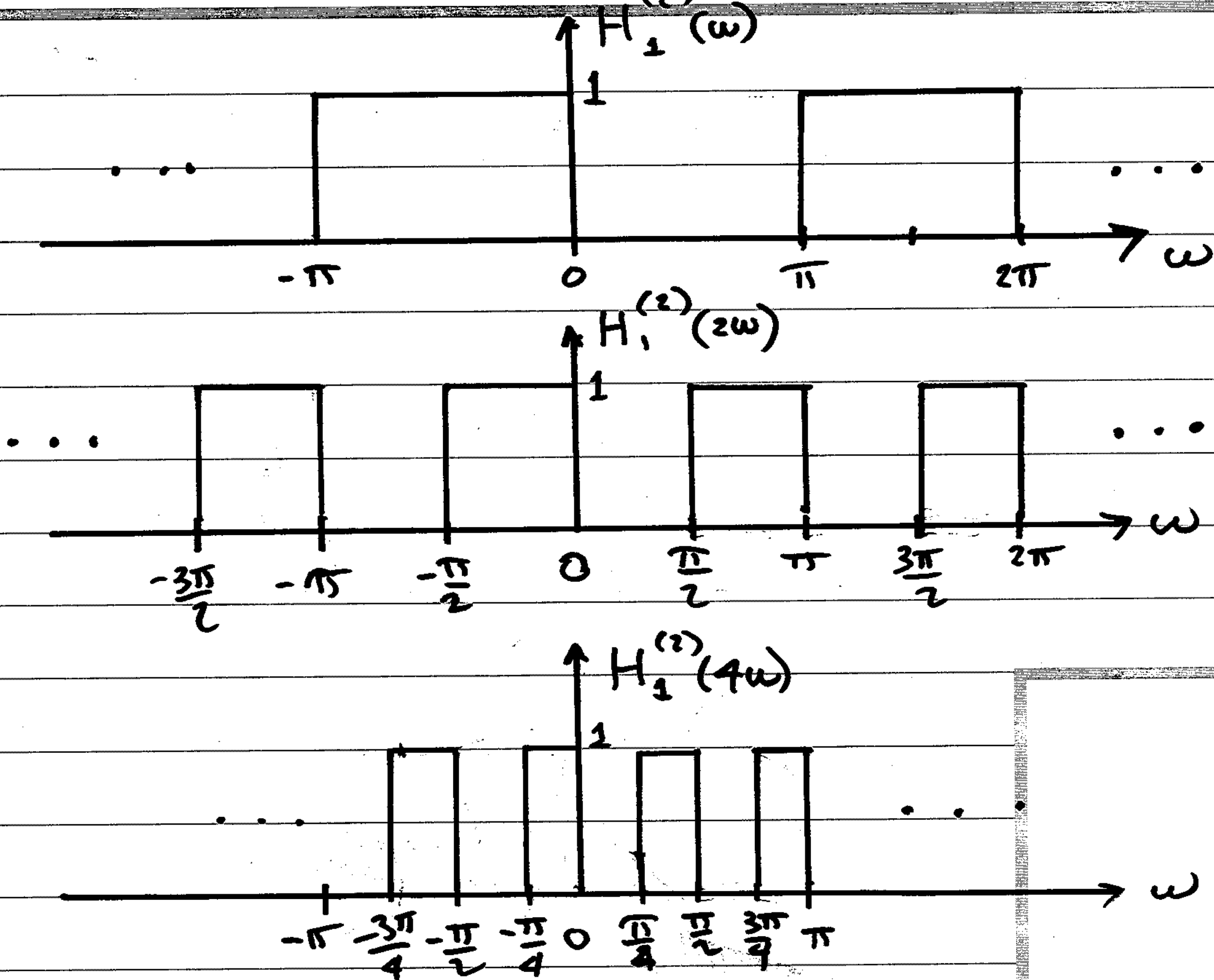
This is a different set of filters than what was on the exam. I changed the problem at the last minute so that you could make use of solutions to old Exam 3's. See Exam 3 F19 for the solution to your Prob. 2.





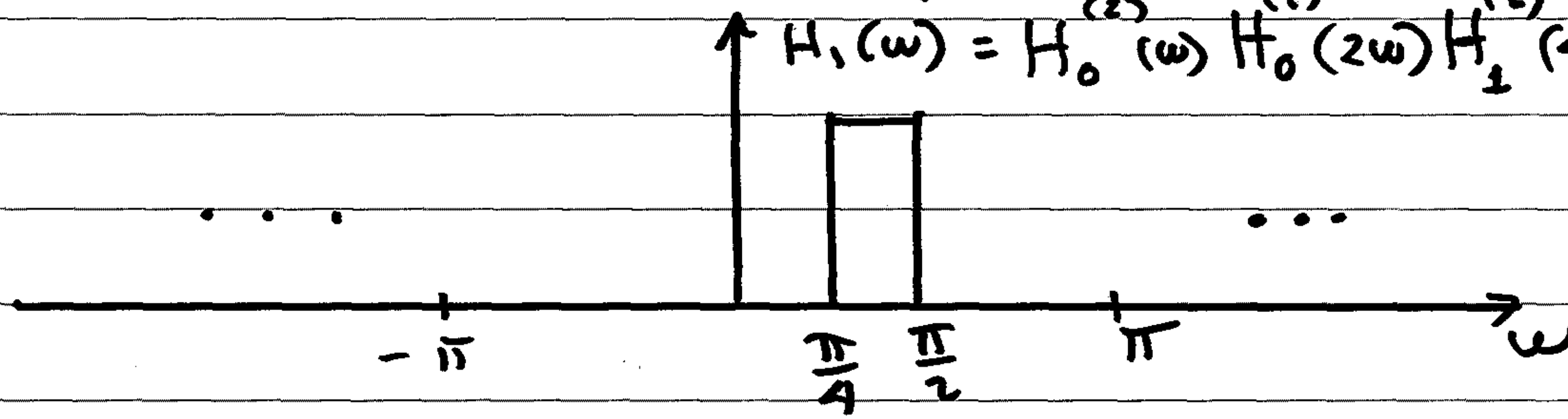


2b


$H_1^{(2)}(\omega)$
1
$-\pi$
0
π
2π
ω
$H_1^{(2)}(2\omega)$
1
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
ω
$H_1^{(2)}(4\omega)$
1
$-\pi$
$-\frac{3\pi}{4}$
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
0
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
ω

Final answer for Prob 2. (2c)

$$H_1(\omega) = H_0^{(2)}(\omega)\, H_0^{(2)}(2\omega)\, H_1^{(2)}(4\omega)$$



Solution to Prob. 3 Condensed (Answer Key)

(i)

(a) $$V_k(\omega) = \frac{1}{4}\sum_{k=0}^{3} H_k\left(\frac{4\omega - k2\pi}{4}\right) X\left(\frac{4\omega - k2\pi}{4}\right)$$

$$\begin{bmatrix} V_0(\omega) \\ V_1(\omega) \\ V_2(\omega) \\ V_3(\omega) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} H_0(\omega) & H_0(\omega - \frac{\pi}{2}) & H_0(\omega - \pi) & H_0(\omega - \frac{3\pi}{2}) \\ H_1(\omega) & H_1(\omega - \frac{\pi}{2}) & H_1(\omega - \pi) & H_1(\omega - \frac{3\pi}{2}) \\ H_2(\omega) & H_2(\omega - \frac{\pi}{2}) & H_2(\omega - \pi) & H_2(\omega - \frac{3\pi}{2}) \\ H_3(\omega) & H_3(\omega - \frac{\pi}{2}) & H_3(\omega - \pi) & H_3(\omega - \frac{3\pi}{2}) \end{bmatrix} \begin{bmatrix} X(\omega) \\ X(\omega - \frac{\pi}{2}) \\ X(\omega - \pi) \\ X(\omega - \frac{3\pi}{2}) \end{bmatrix}$$

$$Y(\omega) = \begin{bmatrix} G_0(\omega) & G_1(\omega) & G_2(\omega) & G_3(\omega) \end{bmatrix} \begin{bmatrix} V_0(\omega) \\ V_1(\omega) \\ V_2(\omega) \\ V_3(\omega) \end{bmatrix}$$

where:

$G_k(\omega) = H_k^*(\omega) \qquad k = 0, 1, 2, 3$

since $g_k[n] = h_k[-n]$

- Two Aspects to Perfect Reconstruction

(a) Distortionless (b) Alias-Free

(a) Check Distortionless:

Require: $\frac{1}{4}\sum_{k=0}^{3} H_k(\omega) H_k^*(\omega) = C \;\forall \omega$ (or: $Ce^{-jn_0\omega}$ more generally)

$C = 30$ in our case here

This, in turn, means/requires:

$$\frac{1}{4}\sum_{k=0}^{3} \underbrace{h_k[n] * h_k^*[n]}_{r_{h_k h_k}[n]} = \underbrace{30}_{C}\,\delta[n]$$

$n=0$ ↓

$r_{h_0h_0}[n] = \{4, 11, 20, 30, 20, 11, 4\}$

$r_{h_1h_1}[n] = \{-6, 11, -18, 30, -18, 11, -6\}$

$r_{h_2h_2}[n] = \{6, -11, -10, 30, -10, -11, 6\}$

$r_{h_3h_3}[n] = \{-4, -11, 8, 30, 8, -11, -4\}$

sum to $4(30)\,\delta[n]$ ⇒ Distortionless!

(ii) Since $G_k(\omega) = H_k^*(\omega)$, then: $G_k(\omega)\, H_k(\omega) = |H_k(\omega)|^2$

$k = 0, 1, 2, 3$

and:

$$r_{h_k h_k}[n] = h_k[n] * h_k^*[-n] \overset{\text{DTFT}}{\longleftrightarrow} G_k(\omega)\, H_k(\omega) = H_k^*(\omega)\, H_k(\omega)$$

also:

$$r_{\underset{\ell\neq}{h_\ell} h_k}[n] = \underset{\ell\neq}{h_\ell}[n] * h_k^*[-n] \overset{\text{DTFT}}{\longleftrightarrow} H_\ell(\omega)\, H_k^*(\omega) = H_k^*(\omega)\, H_\ell(\omega)$$

also:

$$e^{j\ell\frac{2\pi}{4}n} h_k[n] * h_k^*[-n] \overset{\text{DTFT}}{\longleftrightarrow} H_k\left(\omega - \ell\frac{2\pi}{4}\right) H_k^*(\omega)$$

$$Y(\omega) = \begin{bmatrix} H_0^*(\omega) & H_1^*(\omega) & H_2^*(\omega) & H_3^*(\omega) \end{bmatrix} \begin{bmatrix} V_0(\omega) \\ V_1(\omega) \\ V_2(\omega) \\ V_3(\omega) \end{bmatrix}$$

$$e^{j\frac{\pi}{2}n} h_0[n] * h_0^*[n] =$$

$$\{4,\ -10+6j,\ -8-12j,\ -6-10j,\ -3-8j,\ -4j\}$$

$$+\,e^{j\frac{\pi}{2}n} h_1[n] * h_1^*[-n] =$$

$$\{-6,\ 10-4j,\ -12-8j,\ 4+10j,\ -8-3j,\ 6j\}$$

$$+\,e^{j\frac{\pi}{2}n} h_2[n] * h_2^*[n] =$$

$$\{6,\ -10+4j,\ 8+12j,\ -4-10j,\ 3+8j,\ -6j\}$$

$$+\,e^{j\frac{\pi}{2}n} h_3[n] * h_3^*[n] =$$

$$\{-4,\ 10-6j,\ 12+8j,\ 6+10j,\ 8+3j,\ 4j\}$$

$$\sum_{k=0}^{3} \{e^{j\frac{\pi}{2}n} h_k[n]\} * h_k^*[-n] = \{0\}\ \forall n \xleftrightarrow{\text{DTFT}} \sum_{k=0}^{3} H_k(\omega - \tfrac{\pi}{2})\, H_k^*(\omega) = 0$$

(ii-4)

$$e^{j\pi n} h_0[n] * h_0^*[-n] =$$

$$\{4, -5, 8, -10, -8, -5, -4\}$$

$$e^{j\pi n} h_1[n] * h_1^*[-n] =$$

$$\{-6, 5, -10, 10, 10, 5, 6\}$$

$$e^{j\pi n} h_2[n] * h_2^*[-n] =$$

$$\{6, 5, -18, -10, 18, 5, -6\}$$

$$e^{j\pi n} h_3[n] * h_3^*[-n] =$$

$$\{-4, -5, 20, 10, -20, -5, 4\}$$

$$\sum_{k=0}^{3} \{e^{j\pi n} h_k[n]\} * h_k^*[-n] = \{0\} \;\; \forall n \xleftrightarrow{\text{DTFT}} \sum_{k=0}^{3} H_k(\omega-\pi)\, H_k^*(\omega) = 0 \;\; \forall \omega$$

ii-5

$$e^{j\frac{3\pi}{2}n} h_0[n] * h_0^*[-n] =$$

$$\{4,\ 3-8j,\ -10-6j,\ -8+12j,\ -6+10j,\ -3+8j,\ 4j\}$$

$$e^{j\frac{3\pi}{2}n} h_1[n] * h_1^*[-n] =$$

$$\{-6,\ 8-3j,\ 10+4j,\ -12+8j,\ 4-10j,\ -8+3j,\ -6j\}$$

$$e^{j\frac{3\pi}{2}n} h_2[n] * h_2^*[-n] =$$

$$\{6,\ -3+8j,\ -10-4j,\ 8-12j,\ -4+10j,\ 3-8j,\ 6j\}$$

$$e^{j\frac{3\pi}{2}n} h_3[n] * h_3^*[-n] =$$

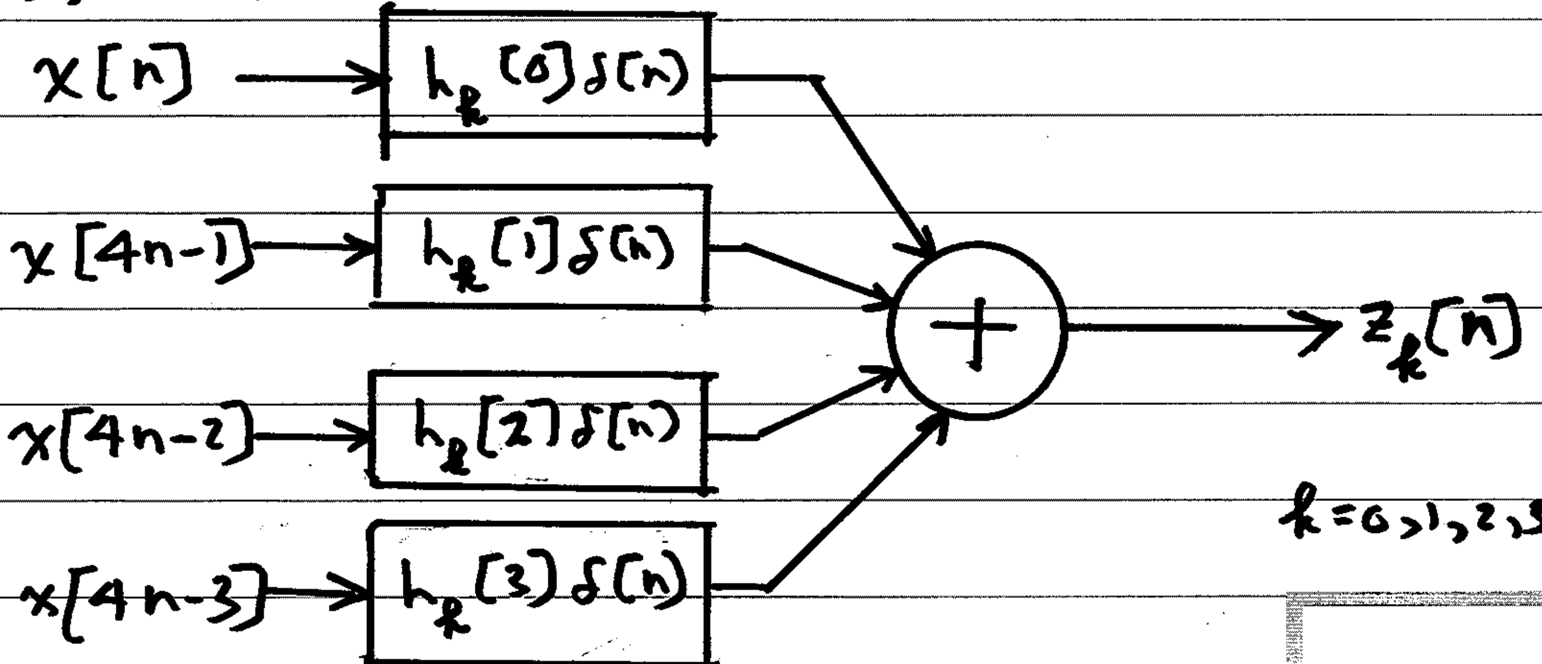
$$\{-4,\ -8+3j,\ 10+6j,\ 12-8j,\ 6-10j,\ 8-3j,\ -4j\}$$

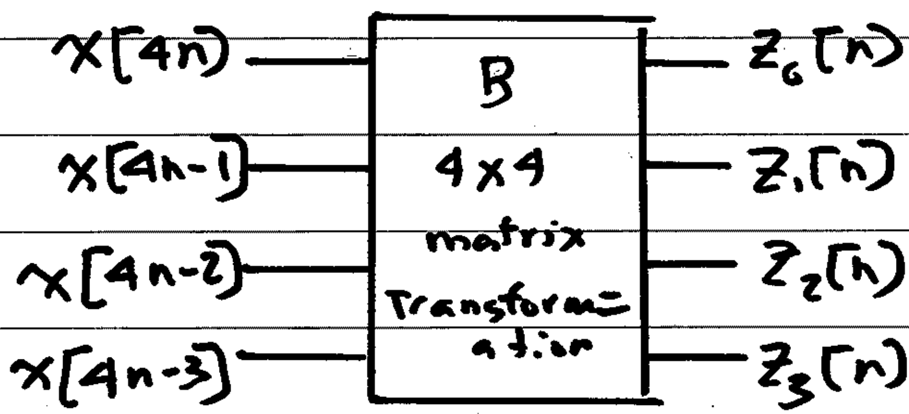
$$\sum_{k=0}^{3} \left\{ e^{j\frac{3\pi}{2}n} h_k[n] \right\} * h_k^*[-n] = \underset{\forall n}{\{0\}} \overset{DTFT}{\longleftrightarrow} \sum_{k=0}^{3} H_k\left(\omega - \tfrac{3\pi}{2}\right) H_k^*(\omega) = 0 \quad \forall \omega$$

(iii) Analysis Side: Efficient Structure

Since each filter of length 4 and decimation by 4:

- Thus, for $k = 0, 1, 2, 3$:





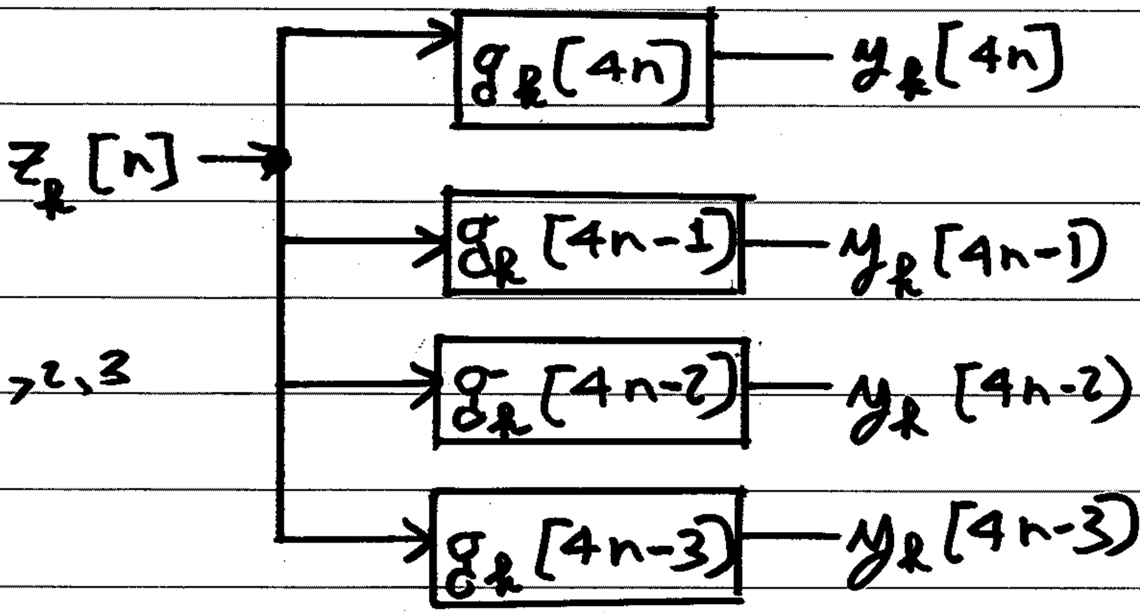
$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ -3 & 4 & 1 & -2 \\ -4 & -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x[4n] \\ x[4n-1] \\ x[4n-2] \\ x[4n-3] \end{bmatrix}$$

(iv) Synthesis Side: Efficient Structure

Note: Given: $g_k[n] = h_k[-n]$

$k=0,1,2,3$ **Thus:** $g_k(n)$ is only nonzero for $n = -3, -2, -1, 0$

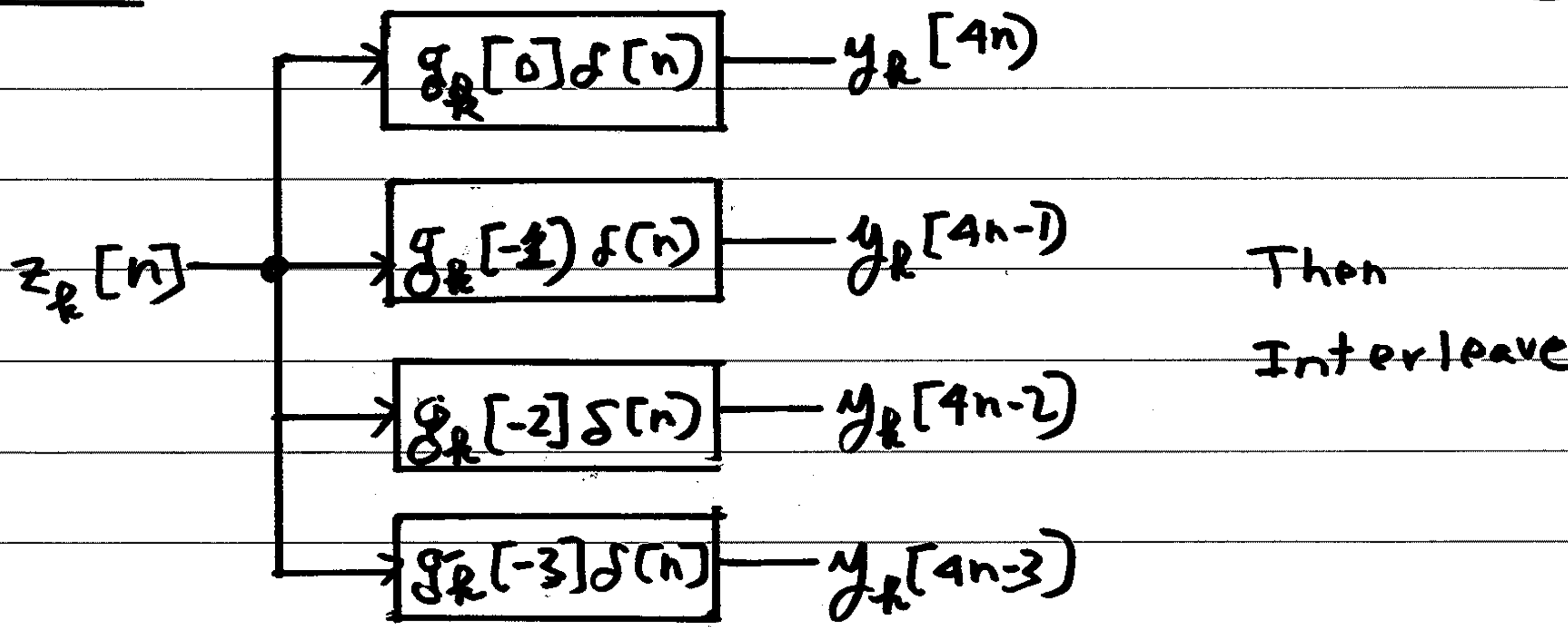
- To match things up with Analysis Side, we will use this version of efficient zero-inserts followed by filtering for $4\uparrow$ "zero inserts":



$k=0,1,2,3$

Then Interleave

Thus: For $k = 0, 1, 2, 3$:



Since $y[n] = \sum_{k=0}^{3} y_k[n]$ and Interleaving is a Linear Operator

$$\begin{bmatrix} y[4n] \\ y[4n-1] \\ y[4n-2] \\ y[4n-3] \end{bmatrix} = \begin{bmatrix} g_0[0] & g_1[0] & g_2[0] & g_3[0] \\ g_0[-1] & g_1[-1] & g_2[-1] & g_3[-1] \\ g_0[-2] & g_1[-2] & g_2[-2] & g_3[-2] \\ g_0[-3] & g_1[-3] & g_2[-3] & g_3[-3] \end{bmatrix} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix}$$

one two

Since: $g_k[n] = h_k[-n] \Rightarrow$

$k = 0, 1, 2, 3$

$$g_k[0] = h_k[0]$$

$$g_k[-1] = h_k[+1]$$

$$g_k[-2] = h_k[+2]$$

$$g_k[-3] = h_k[+3]$$

$k = 0, 1, 2, 3$

(iv)-3

Thus:

$$\begin{bmatrix} y[4n] \\ y[4n-1] \\ y[4n-2] \\ y[4n-3] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & 4 & -3 \\ 3 & -4 & 1 & 2 \\ 4 & 3 & -2 & 1 \end{bmatrix}}_{A = B^T} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = 31 \begin{bmatrix} x[4n] \\ x[4n-1] \\ x[4n-2] \\ x[4n-3] \end{bmatrix}$$

- Since B contains mutually orthogonal rows

$$AB = B^T B = 30\, I$$

(I: 4×4)

- PRFB: $y[n] = 30\, x[n]$