Cover Sheet

Test Duration: 60 minutes.
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed.
Calculators NOT allowed.
This test contains two problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.
Digital Signal Processing I  Exam 3  Fall 2013

Problem 1. [50 pts] For all parts of this problem, let $x[n]$ be a discrete-time rectangular pulse of length $L = 8$ and $h[n]$ be a discrete-time rectangular pulse of length $M = 4$ as defined below:

$$x[n] = u[n] - u[n - 8] \quad \quad h[n] = u[n] - u[n - 4]$$

Also, for all parts of this problem, the reconstructed spectrum is computed according to Equation 1 below:

$$Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \sin \left[ \frac{N}{2} \left( \frac{\omega - 2\pi k}{N} \right) \right] e^{-j \frac{N}{2} \left( \omega - 2\pi k \right)}$$  \hspace{1cm} (1)

(a)-(i) With $X_N(k)$ computed as the 16-pt DFT of $x[n]$ and $H_N(k)$ computed as the 16-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 16$ values of the 16-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(a)-(ii) The product sequence, $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 16$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

(b)-(i) With $X_N(k)$ computed as the 12-pt DFT of $x[n]$ and $H_N(k)$ computed as the 12-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 12$ values of the 12-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(b)-(ii) The product sequence $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 12$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

(c)-(i) With $X_N(k)$ computed as the 8-pt DFT of $x[n]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 8$ values of the 8-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(c)-(ii) The product sequence $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 8$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.
(a)-(i) \( y[n] = x[n] * h[n] \) is of length \( L + M - 1 = 8 + 4 - 1 = 11 \)

\[ y[n] = \left\{ \begin{array}{c}
1, 2, 3, 4, 4, 4, 4, 3, 2, 1 \\
\end{array} \right\}_{n=0}^{11} \]

Since \( N = 16 > 11 \):
\[ y_{16}^N = \{ y[n], 0, 0, 0, 0, 0 \} \]
5 zeroes

(a)-(ii) Perfect reconstruction of \( Y(\omega) = \mathcal{DFT}(y[n]) \)

\[ Y_r(\omega) = H(\omega) \times X(\omega) = \]
\[ = \frac{\sin \left( \frac{8}{2} \omega \right)}{\sin \left( \frac{1}{2} \omega \right)} e^{-j \left( \frac{4}{2} \omega \right)} e^{-j \left( \frac{8-1}{2} \omega \right)} \]

(b)-(i) \( y_{12}[n] = \{ y[n] \}_{0}^{10} \) linear convolution

(b)-(ii) \( Y_r(\omega) = Y(\omega) \) same answer as (a)-(ii)

(c)-(i) \( 11 - 8 = 3 \) pts at end aliased into \( 3 \) pts at beginning

\[ y_8[n] = y[n] + y[n+8] \]
\[ = \{ 4, 4, 4, 4, 4, 4, 4, 4 \} \]

(c)-(ii) \( Y_r(\omega) = 4 \frac{\sin \left( \frac{8}{2} \omega \right)}{\sin \left( \frac{1}{2} \omega \right)} e^{-j \left( \frac{8-1}{2} \omega \right)} \)
Problem 2. [50 points] Consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \sin\left[\frac{5\pi}{8} (n + \ell 8)\right] \frac{\sin\left[\frac{3\pi}{8} (n + \ell 8)\right]}{\pi (n + \ell 8)} \{u[n] - u[n - 8]\}$$

(a) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \leq k \leq 7$. List the values clearly: $H_8(0) = ?$, $H_8(1) = ?$, $H_8(2) = ?$, $H_8(3) = ?$, $H_8(4) = ?$, $H_8(5) = ?$, $H_8(6) = ?$, $H_8(7) = ?$.

(b) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk \frac{2\pi}{8} n} \{u[n] - u[n - 8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h_p[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

Next, consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{8 \sin\left[\frac{5\pi}{8} (n + \ell 8)\right]}{3} \frac{\sin\left[\frac{3\pi}{8} (n + \ell 8)\right]}{\pi (n + \ell 8)} \{u[n] - u[n - 8]\}$$

(c) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \leq k \leq 7$. List the values clearly: $H_8(0) = ?$, $H_8(1) = ?$, $H_8(2) = ?$, $H_8(3) = ?$, $H_8(4) = ?$, $H_8(5) = ?$, $H_8(6) = ?$, $H_8(7) = ?$.

(d) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk \frac{2\pi}{8} n} \{u[n] - u[n - 8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h_p[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.
From the theory of time-domain aliasing

\[
\sum_{\ell = -\infty}^{\infty} x[n + \ell N] \{u[n] - u[n-N]\} \overset{\text{DFT}}{\rightarrow} X(\omega) \quad \omega = \frac{2\pi k}{N}
\]

where: \( x[n] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) \)

This is true even if \( x[n] \) is defined over \(-\infty < n < \infty\)

\((a)\)

\[
h[n] = \frac{\sin\left(\frac{5\pi}{8} n\right)}{\pi n} \overset{\text{DTFT}}{\rightarrow} H(\omega)
\]

Thus:

\[
\sum_{\ell = -\infty}^{\infty} h[n + \ell N] \{u[n] - u[n-N]\} \overset{\text{DFT}}{\rightarrow} H(\omega) \quad \omega_{k} = \frac{2\pi k}{8}
\]

\( k = 0, 1, \ldots, 7 \)

\( H_{8}(0) = H(\frac{\pi}{4}) = 1 \)

\( H_{8}(1) = H(\frac{\pi}{2}) = 1 \)

\( H_{8}(2) = H(\frac{3\pi}{4}) = 1 \)

\( H_{8}(3) = H(\frac{\pi}{8}) = 0 \)

\( H_{8}(4) = H(\frac{3\pi}{8}) = 0 \)

\( H_{8}(5) = H(\frac{5\pi}{8}) = 0 \)

\( H_{8}(6) = H(\frac{7\pi}{8}) = 1 \)

\( H_{8}(7) = H(\frac{3\pi}{4}) = 1 \)

\((b)\) As proved in class:

\[
e^{j\frac{2\pi}{N} n} \{u[n] - u[n-N]\} \overset{\text{N}}{\rightarrow} h[n] = H(\frac{2\pi k}{N}) e^{j\frac{2\pi k}{8} n}
\]

Thus:

\[
y[n] = \sum_{k=0}^{2} e^{j\frac{2\pi k}{8} n} + \sum_{k=6}^{7} e^{j\frac{2\pi k}{8} n} \{u[n] - u[n-N]\}
\]
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Supporting Math Development From Class Lecture

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\[
\sum_{l=-\infty}^{\infty} x[n + lN] = x[n] \ast \sum_{l=-\infty}^{\infty} s[n + lN] = x[n] \ast \sum_{k=0}^{N-1} e^{j \frac{2\pi}{N} k n}
\]

\[
\leftrightarrow X(\omega) \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta(\omega - k \frac{2\pi}{N})
\]

\[
= \sum_{k=0}^{N-1} \frac{2\pi}{N} X(k \frac{2\pi}{N}) \delta(\omega - k \frac{2\pi}{N})
\]

Recall: \( u(n) - u(n-N) \leftrightarrow R(\omega) = \frac{\sin \left( \frac{N}{2} \omega \right)}{\sin \left( \frac{1}{2} \omega \right)} e^{-j \frac{N-1}{2} \omega} \)

Define: \( P(\omega) = \frac{1}{N} R(\omega) \)

Recall: \( e^{jw_0n} \leftrightarrow DTFT \sum_{k} \delta(\omega - w_0)
\)

Recall: \( x[n] y[n] \leftrightarrow DTFT \frac{1}{2\pi} \int X(\omega) \ast Y(\omega) \)

\[
\sum_{l=-\infty}^{\infty} x[n + lN] \{ u[n] - u[n-N] \} \leftrightarrow \sum_{k=0}^{N-1} X(k \frac{2\pi}{N}) \delta(\omega - k \frac{2\pi}{N}) \ast P(\omega)
\]

\[
\leftrightarrow x_p[n] \leftrightarrow X_p(\omega) = \sum_{k=0}^{N-1} X(k \frac{2\pi}{N}) P(\omega - k \frac{2\pi}{N})
\]

\[
X_p[n] \xrightarrow{DFT} X_p(\omega) \bigg|_{\omega = k \frac{2\pi}{N}, k = 0, 1, \ldots, N-1}
\]

\( P(\omega) : P(0) = 1 \quad P(\frac{2\pi}{N}) = 0 \text{ for } k \neq 0 \)

\( P(\omega - k \frac{2\pi}{N}) = 1 \text{ at } \omega = k \frac{2\pi}{N} \Rightarrow = 0 \text{ for } l \frac{2\pi}{N} \neq k \)

where one peaks, all the others cross through zero

Thus: \( X_p[n] \xrightarrow{DFT} X(k \frac{2\pi}{N}), k = 0, 1, \ldots, N-1 \quad \text{(peak value = 1)} \)