SOLUTION

NAME: 2013
Digital Signal Processing I **Exam 3** **Fall 2013**
Session 41 **25 Nov. 2013**

Cover Sheet

Test Duration: 60 minutes.
Open Book but Closed Notes. One 8.5 x 11 crib sheet allowed
Calculators NOT allowed.
This test contains **two** problems.
All work should be done on the blank pages provided.
Your answer to each part of the exam should be clearly labeled.

Digital Signal Processing I   Exam 3   Fall 2013

Problem 1. [50 pts] For all parts of this problem, let $x[n]$ be a discrete-time rectangular pulse of length $L = 8$ and $h[n]$ be a discrete-time rectangular pulse of length $M = 4$ as defined below:

$$x[n] = u[n] - u[n-8] \qquad\qquad h[n] = u[n] - u[n-4]$$

Also, for all parts of this problem, the reconstructed spectrum is computed according to Equation 1 below:

$$Y_r(\omega) = \sum_{k=0}^{N-1} Y_N(k) \frac{\sin\left[\frac{N}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]}{N \sin\left[\frac{1}{2}\left(\omega - \frac{2\pi k}{N}\right)\right]} e^{-j\frac{N-1}{2}\left(\omega - \frac{2\pi k}{N}\right)} \tag{1}$$

(a)-(i) With $X_N(k)$ computed as the 16-pt DFT of $x[n]$ and $H_N(k)$ computed as the 16-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 16$ values of the 16-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(a)-(ii) The product sequence, $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 16$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

(b)-(i) With $X_N(k)$ computed as the 12-pt DFT of $x[n]$ and $H_N(k)$ computed as the 12-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 12$ values of the 12-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(b)-(ii) The product sequence $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 12$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

(c)-(i) With $X_N(k)$ computed as the 8-pt DFT of $x[n]$ and $H_N(k)$ computed as the 8-pt DFT of $h[n]$, the product $Y_N(k) = X_N(k)H_N(k)$ is formed. Determine the $N = 8$ values of the 8-pt Inverse DFT of $Y_N(k) = X_N(k)H_N(k)$.

(c)-(ii) The product sequence $Y_N(k) = X_N(k)H_N(k)$, formed as directly above with $N = 8$, is used in Eqn (1). Write a closed-form expression for the reconstructed spectrum $Y_r(\omega)$.

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(a)-(i) $y[n] = x[n] * h[n]$ is of length $L+M-1 = 8+4-1 = 11$

$$y[n] = \{\underset{\underset{n=0}{\uparrow}}{1}, 2, 3, 4, 4, 4, 4, 4, 3, 2, 1\}$$

Since $N = 16 > 11$: $y_{16}[n] = \{y[n], \underbrace{0, 0, 0, 0, 0}_{5 \text{ zeroes}}\}$

(a)-(ii) Perfect reconstruction of $Y(\omega) = \text{DTFT}\{y[n]\}$

$$Y_r(\omega) = H(\omega) X(\omega) =$$

$$= \frac{\sin\left(\frac{8}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(8-1)}{2}\omega} \frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(4-1)}{2}\omega}$$

(b)-(i) $y_{12}[n] = \{y[n], 0\}$

linear convolution

(b)-(ii) $Y_r(\omega) = Y(\omega)$ same answer as (a)-(ii)

(c)-(i) $11 - 8 = 3$ pts at end aliased into 3 pts at beginning

$$y_8[n] = y[n] + y[n+8]$$
$$= \{\underset{\uparrow}{4}, 4, 4, 4, 4, 4, 4, 4\}$$

(c)-(ii)

$$Y_r(\omega) = 4 \frac{\sin\left(\frac{8}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(8-1)}{2}\omega}$$

Problem 2. [50 points] Consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin\left[\frac{5\pi}{8}(n + \ell 8)\right]}{\pi(n + \ell 8)} \{u[n] - u[n-8]\}$$

(a) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \le k \le 7$. List the values clearly: $H_8(0) =?$, $H_8(1) =?$, $H_8(2) =?$, $H_8(3) =?$, $H_8(4) =?$, $H_8(5) =?$, $H_8(6) =?$, $H_8(7) =?$.

(b) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h_p[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

Next, consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{8}{3} \frac{\sin\left[\frac{5\pi}{8}(n + \ell 8)\right]}{\pi(n + \ell 8)} \frac{\sin\left[\frac{3\pi}{8}(n + \ell 8)\right]}{\pi(n + \ell 8)} \{u[n] - u[n-8]\}$$

(c) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \le k \le 7$. List the values clearly: $H_8(0) =?$, $H_8(1) =?$, $H_8(2) =?$, $H_8(3) =?$, $H_8(4) =?$, $H_8(5) =?$, $H_8(6) =?$, $H_8(7) =?$.

(d) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h_p[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

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From the theory of time-domain aliasing

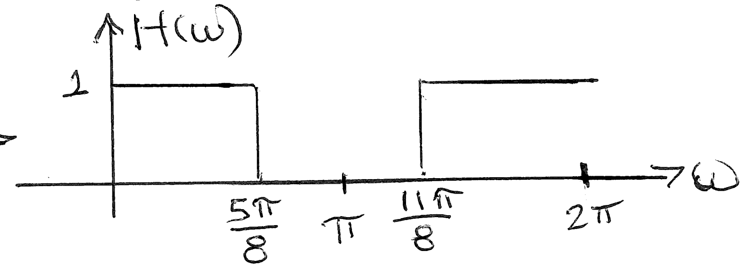
$$\sum_{\ell=-\infty}^{\infty} x[n+\ell N]\{u[n]-u[n-N]\} \overset{DFT}{\underset{N}{\longleftrightarrow}} X(\omega)\Big|_{\omega_k=\frac{2\pi k}{N}}$$

$k=0,1,\ldots,N-1$

where: $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

This is true even if $x[n]$ is defined over $-\infty<n<\infty$

(a) $h[n]=\dfrac{\sin\left(\frac{5\pi}{8}n\right)}{\pi n} \overset{DTFT}{\longleftrightarrow}$



Thus:

$$\sum_{\ell=-\infty}^{\infty} h[n+\ell N]\{u[n]-u[n-8]\} \overset{DFT}{\underset{8}{\longleftrightarrow}} H(\omega)\Big|_{\omega_k=k\frac{2\pi}{8}}$$

$k=0,1,\ldots,7$

$H_8(0)=1 \quad H_8(1)=H\left(\frac{2\pi}{8}\right)=1 \quad H_8(2)=H\left(\frac{4\pi}{8}\right)=1$

$H_8(3)=H\left(\frac{6\pi}{8}\right)=0 \quad H_8(4)=H\left(\frac{8\pi}{8}\right)=0$

$H_8(5)=H\left(\frac{10\pi}{8}\right)=0 \quad H_8(6)=H\left(\frac{12\pi}{8}\right)=1$

$H_8(7)=H\left(\frac{14\pi}{8}\right)=1$

(b) As proved in class:

$$e^{jk\frac{2\pi}{N}n}\{u[n]-u[n-N]\} \; (N) \; h[n] = H\left(k\frac{2\pi}{N}\right)e^{jk\frac{2\pi}{8}n}\cdot(u[n]-u[n-N])$$

Thus:

$$y[n]=\left\{\sum_{k=0}^{2} e^{jk\frac{2\pi}{8}n}+\sum_{k=6}^{7} e^{jk\frac{2\pi}{8}n}\right\}\cdot(u[n]-u[n-8])$$

$$\frac{5\pi}{8} - \frac{3\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$$

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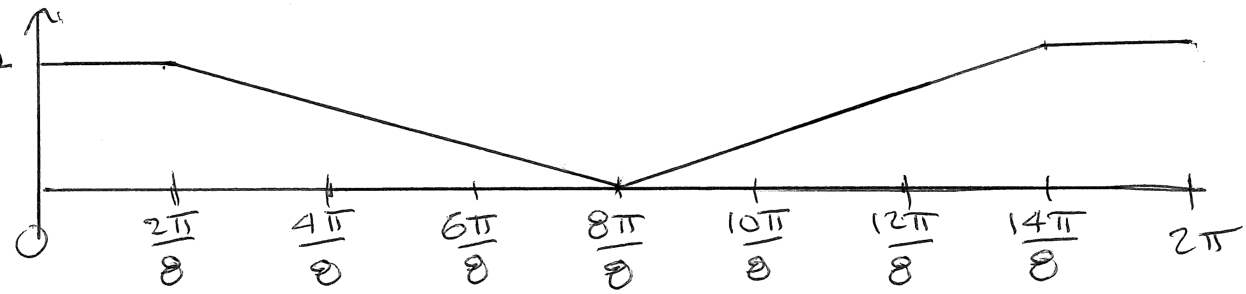
$$h_0[n] = \frac{8}{3}\,\frac{\sin\left(\frac{5\pi}{8}n\right)}{\pi n}\,\frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \xleftrightarrow{\text{DTFT}}$$



$H_8(0) \rightarrow$

$$y[n] = 1\, e^{j0n}\,\{u[n]-u[n-8]\}$$

$$+\ 1\, e^{j\frac{2\pi}{8}n} \quad " \qquad H_8(1)$$

$$+\ \frac{2}{3}\, e^{j\frac{4\pi}{8}n} \quad " \qquad H_8(2)$$

$$+\ \frac{1}{3}\, e^{j\frac{6\pi}{8}n} \quad " \qquad H_8(3)$$

$$+\ 0\, e^{j\frac{8\pi}{8}n} \quad " \qquad H_8(4)$$

$$+\ \frac{1}{3}\, e^{j\frac{10\pi}{8}n} \quad " \qquad H_8(5)$$

$$+\ \frac{2}{3}\, e^{j\frac{12\pi}{8}n} \quad " \qquad H_8(6)$$

$$+\ 1\, e^{j\frac{14\pi}{8}n} \quad " \qquad H_8(7)$$

$$H_8(0) = H(0) = 1$$

$$H_8(1) = H\left(\frac{2\pi}{8}\right) = 1$$

$$H_8(2) = H\left(\frac{4\pi}{8}\right) = \frac{2}{3}$$

$$H_8(3) = H\left(\frac{6\pi}{8}\right) = \frac{1}{3}$$

$$H_8(4) = H\left(\frac{8\pi}{8}\right) = 0$$

$$H_8(5) = H\left(\frac{10\pi}{8}\right) = \frac{1}{3}$$

$$H_8(6) = H\left(\frac{12\pi}{8}\right) = \frac{2}{3}$$

$$H_8(7) = H\left(\frac{14\pi}{8}\right) = 1$$

Supporting Math Development From Class Lecture

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$$\sum_{l=-\infty}^{\infty} x[n+lN] = x[n] * \sum_{l=-\infty}^{\infty} \delta[n+lN] = x[n] * \sum_{k=0}^{N-1} \frac{1}{N} e^{jk\frac{2\pi}{N}n}$$

$$\xleftrightarrow{DTFT} X(\omega) \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta\left(\omega - k\frac{2\pi}{N}\right)$$

$$= \sum_{k=0}^{N-1} \frac{2\pi}{N} X\left(k\frac{2\pi}{N}\right) \delta\left(\omega - k\frac{2\pi}{N}\right)$$

Recall: $u[n] - u[n-N] \xleftrightarrow{DTFT} R(\omega) = \dfrac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(N-1)}{2}\omega}$

Define: $P(\omega) = \frac{1}{N} R(\omega)$ Recall: $e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\delta(\omega - \omega_0)$

Recall: $x[n]\, y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} X(\omega) * Y(\omega)$

$$\underbrace{\sum_{l=-\infty}^{\infty} x[n+lN]\{u[n]-u[n-N]\}}_{x_p[n]} \xleftrightarrow{DTFT} \sum_{k=0}^{N-1} X\left(k\frac{2\pi}{N}\right)\delta\left(\omega - k\frac{2\pi}{N}\right) * P(\omega)$$

$$x_p[n] \xleftrightarrow{DTFT} X_p(\omega) = \sum_{k=0}^{N-1} X\left(k\frac{2\pi}{N}\right) P\left(\omega - k\frac{2\pi}{N}\right)$$

$$x_p[n] \xleftrightarrow[N]{DFT} X_p(\omega)\Big|_{\omega = k\frac{2\pi}{N}},\ k = 0, 1, \ldots, N-1$$

$P(\omega)$: $P(0) = 1$ $P\left(k\frac{2\pi}{N}\right) = 0$ for $k \neq 0$

$P\left(\omega - k\frac{2\pi}{N}\right) = 1$ at $\omega = k\frac{2\pi}{N}$ $\Rightarrow = 0$ for $l\frac{2\pi}{N}$ $l \neq k$

where one peaks, all the others cross through zero (peak value $= 1$)

Thus: $x_p[n] \xleftrightarrow[N]{DFT} X\left(k\frac{2\pi}{N}\right),\ k = 0, 1, \ldots, N-1$