

**NAME:**  
**ECE 538 Digital Signal Processing I**

**Exam 3**  
**26 Nov. 2012**

## **Cover Sheet**

**WRITE YOUR NAME ON THIS COVER SHEET**

Test Duration: 60 minutes.

Open Book but Closed Notes.

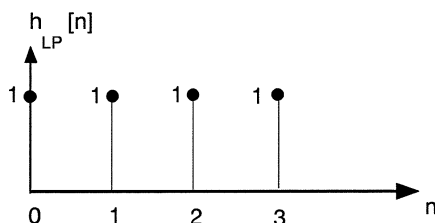
Calculators NOT allowed.

All work should be done in the space provided.

**Problem 1.** This problem is about determining whether the four-band analysis filter bank and corresponding four-band synthesis filter bank in Figure 1 achieves Perfect Reconstruction when the respective impulse responses for the four filters

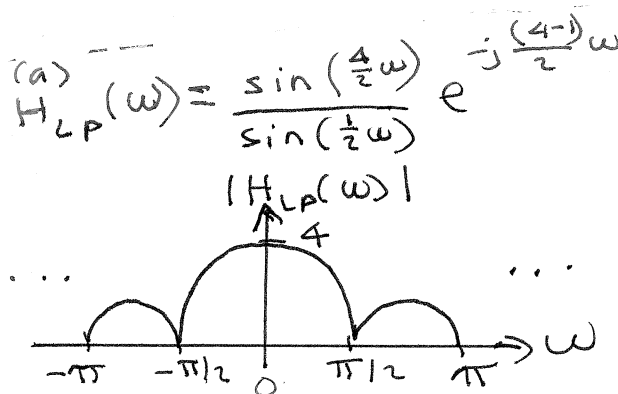
$$h_k[n] = e^{j(k\frac{2\pi}{4})n} h_{LP}[n], \quad k = 0, 1, 2, 3$$

are defined in terms of the crude lowpass filter with impulse response  $h_{LP}[n] = u[n] - u[n-4]$



To answer this question, we analyze the efficient version of the overall filter bank in Fig. 2.

- (a) Plot the magnitude of the DTFT  $H_{LP}(\omega)$  of  $h_{LP}[n]$  over  $-\pi < \omega < \pi$ . Show as much detail as possible. Will there be aliasing present in each of the outputs  $z_k[n]$ ?



Since  $h_{LP}[n]$  passes substantial energy outside the band  $|\omega| < \frac{\pi}{4}$ , there will be substantial aliasing.

# Problem 1.

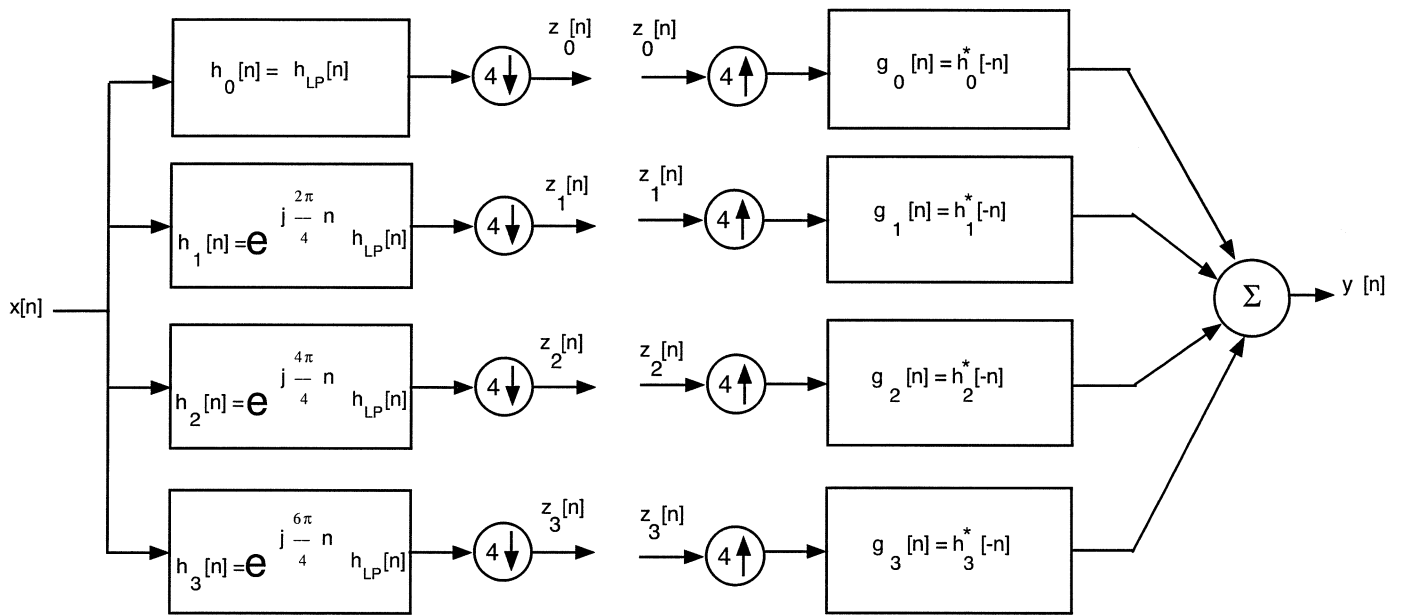


Figure 1.

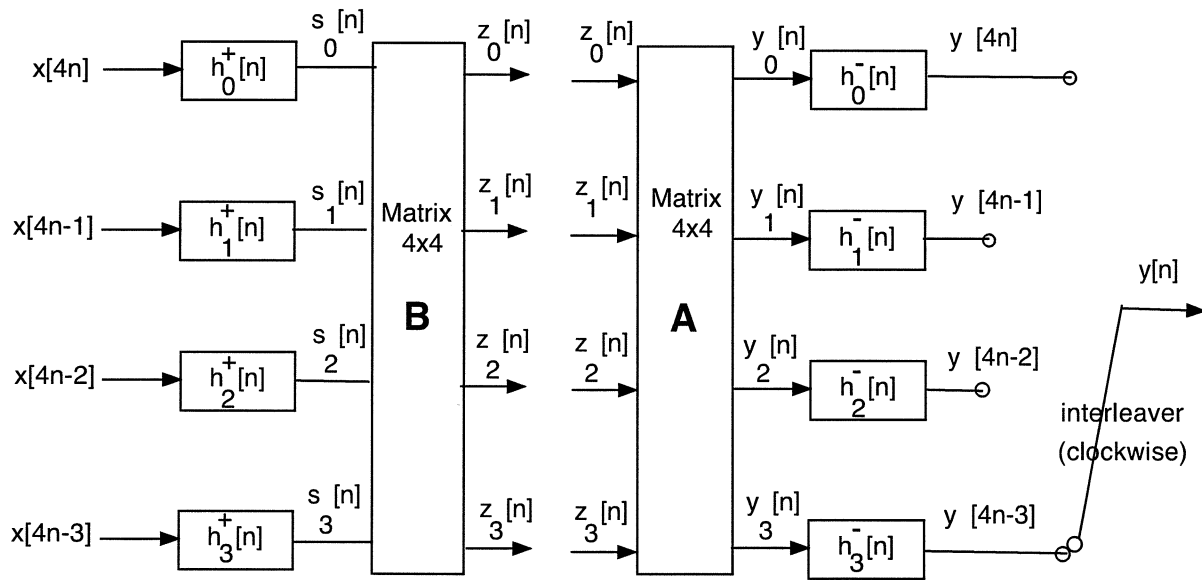
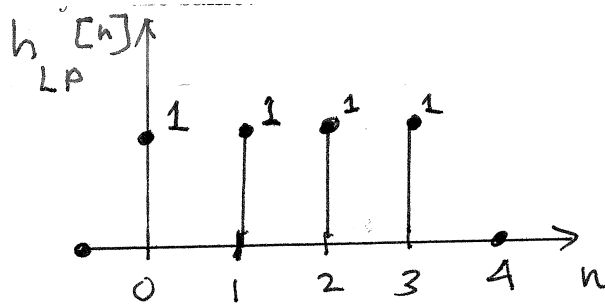
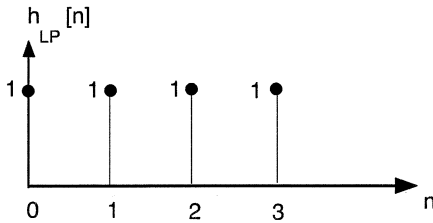


Figure 2.

(b) The four polyphase components of  $h_{LP}[n]$  on the left side of Figure 2 are defined as

$$h_\ell^+[n] = h_{LP}[4n + \ell], \quad \ell = 0, 1, 2, 3. \quad (1)$$

where  $h_{LP}[n] = u[n] - u[n-4]$  as defined previously and plotted below. Do a separate stem plot of each of the four impulse responses  $h_\ell^+[n] = h_{LP}[4n + \ell]$ ,  $\ell = 0, 1, 2, 3$ .



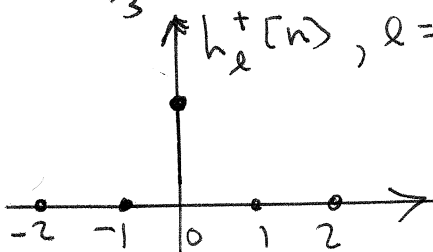
(b)

$$h_0^+[n] = h_{LP}[4n] = \delta[n] \quad (\text{every 4th value starting at 0})$$

$$h_1^+[n] = h_{LP}[4n+1] = \delta[n] \quad (\text{e.g. } h_1^+[0] = h_{LP}[1])$$

$$h_2^+[n] = h_{LP}[4n+2] = \delta[n] \quad (\text{e.g. } h_2^+[0] = h_{LP}[2])$$

$$h_3^+[n] = h_{LP}[4n+3] = \delta[n] \quad (\text{e.g. } h_3^+[0] = h_{LP}[3])$$



a) the same

- (c) The digital subbanding on the left hand side of Figure 1 is implement efficiently with the polyphase filters defined in part (b) via the block diagram on the left side of Figure 2, where the respective input signals are defined as below.

Fill in the values of  $\beta_{k\ell}$  in the table below so that the outputs,  $z_k[n]$ ,  $k = 0, 1, 2, 3$ , at the output of the left hand side in Figure 2 are exactly the same as the outputs  $z_k[n]$  of the left hand side in Figure 1. **Show all work below the table and on the next page. If you simply fill in numbers with no work, you will get no credit.**

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \beta_{11} & \beta_{12} & \beta_{13} \\ 1 & \beta_{21} & \beta_{22} & \beta_{23} \\ 1 & \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} \Rightarrow \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \beta_{11} & \beta_{12} & \beta_{13} \\ 1 & \beta_{21} & \beta_{22} & \beta_{23} \\ 1 & \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \quad (2)$$

$\beta_{k\ell}$	$\ell=1$	$\ell=2$	$\ell=3$
$k=1$	$j$	$-1$	$-j$
$k=2$	$-1$	$1$	$-1$
$k=3$	$-j$	$-1$	$j$

polyphase components:

$$f_k[4n+l] = e^{j\frac{2\pi}{4}kl(4n+l)} h_{LP}[4n+l]$$

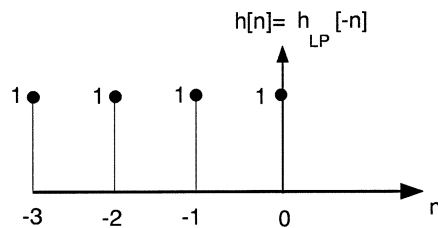
$$= e^{j2\pi kn} e^{j\frac{\pi}{2}kl} h_L^+[n]$$

$$= (1) (j)^{kl} h_L^+[n]$$

scalars are implemented after the filters invoking linearity

$$\left. \begin{array}{l} \text{for } k=1: (j)^l, \quad l=0,1,2,3 \\ \text{for } k=2: (j^2)^l = (-1)^l, \quad l=1,2,3 \\ \text{for } k=3: (j^3)^l = (-j)^l, \quad l=1,2,3 \end{array} \right\} \text{see table above}$$

(d) The “zero inserts followed by filtering” operations on the right side of Figure 1 are implemented efficiently in a collective manner via the right side of Figure 2, where  $h_\ell^-[n] = h[Ln - \ell]$  are the polyphase components of  $h[n] = g_0[n] = f_0^*[-n] = h_{LP}[-n]$ :

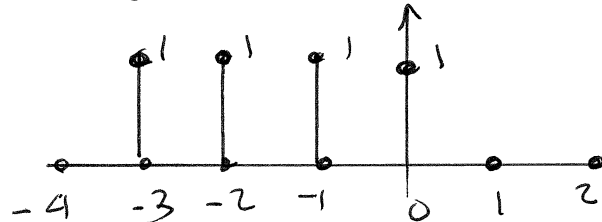


Again, the following four polyphase components of  $h[n] = g_0[n] = h_{LP}[-n]$ , plotted above, are

$$h_\ell^-[n] = h[4n - \ell], \quad \ell = 0, 1, 2, 3. \quad (3)$$

Do a separate stem plot of each of the four impulse responses  $h_\ell^-[n]$ ,  $\ell = 0, 1, 2, 3$ .

where:  $g_0[n] = h_{LP}[-n]$



$h_0^-[n] = \delta[n]$  every 4<sup>th</sup> value of  $g_0[n]$  starting at 0

$h_1^-[n] = \delta[n]$  e.g.:  $h_1^-[0] = g_0[-1]$

$\Rightarrow$  every 4<sup>th</sup> value starting at -1

$h_2^-[n] = g_0[4n-2]$   
 $= \delta[n]$

e.g.:  $h_2^-[0] = g_0[-2]$   
every 4<sup>th</sup> value starting at -2

$h_3^-[n] = g_0[4n-3]$   
 $= \delta[n]$

e.g.:  $h_3^-[0] = g_0[-3]$   
every 4<sup>th</sup> value starting at -3

- (e) Determine the value of each of the amplitude coefficients  $\alpha_{\ell k}$ ,  $k = 1, 2, 3$ ,  $\ell = 1, 2, 3$  so that the clockwise-interleaved output  $y[n]$  in Figure 2 is the SAME as the overall output in Figure 1. Show all work on this and the next page but input your final answers in the table below.

$$\begin{bmatrix} y_0[n] \\ y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 1 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 1 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 1 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \quad (4)$$

$\alpha_{\ell k}$	$k=1$	$k=2$	$k=3$
$\ell=1$	$-j$	$-1$	$j$
$\ell=2$	$-1$	$1$	$-1$
$\ell=3$	$j$	$-1$	$-j$

"negative" polyphase components:  $g_k[4n-l]$

where:  $g_k[n] = h_k^*[-n]$

recall:  $f_k[n] = e^{j \frac{2\pi}{4} kn} h_{LP}[n]$

$$\begin{aligned} \Rightarrow g_k[n] &= e^{-j \frac{2\pi}{4} k(-n)} h_{LP}[-n] \\ &= e^{j \frac{2\pi}{4} kn} h_{LP}[-n] \end{aligned}$$

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$$\text{Thus: } g_k[4n-l] = e^{j \frac{2\pi}{4} (4n-l)} h_{LP}[-(4n-l)]$$

$$= \underbrace{e^{j 2\pi kn}}_{=1} \underbrace{e^{-j \frac{\pi}{2} kl}}_{\text{conjugate of answers to part (c)}} \underbrace{h_{LP}[l-4n]}_{(d)}$$

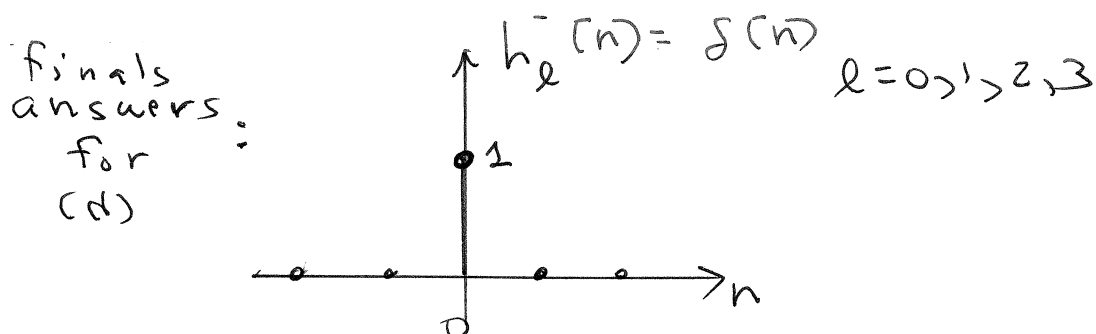
conjugate  
of answers  
to part (c)

Thus, the  
matrix A  
is just the  
conjugate of  
the matrix B

See table answers  
on previous page



(f) Does  $y[n] = 4x[n]$ ? Justify your answer with all that you've developed so far for parts (a) through (e). **Hint:** assess if  $y[4n - \ell] = 4x[4n - \ell]$ , for  $\ell = 0, 1, 2, 3$ .



(f)  
matrix B is 4-pt inverse DFT matrix, except for scaling  $1/4$

matrix A is 4-pt DFT matrix

One can verify:  $\underline{B} \underline{A} = 4 \underline{I}$   
 $\uparrow$   
 $4 \times 4$  Identity Matrix

In addition:

$$\underline{h}_\ell^+ [n] * \underline{h}_\ell^- [n] = \delta[n] * \delta[n] = \delta[n]$$

in series since  
 cascade of B and A  
 is Identity Transformation ( $\times 4$ )

Thus:  $y[4n - \ell] = 4x[4n - \ell]$ ,  $\ell = 0, 1, 2, 3$

Hence:  $y[n] = 4x[n]$  perfect reconstruction!

**Problem 2.** [20 points]

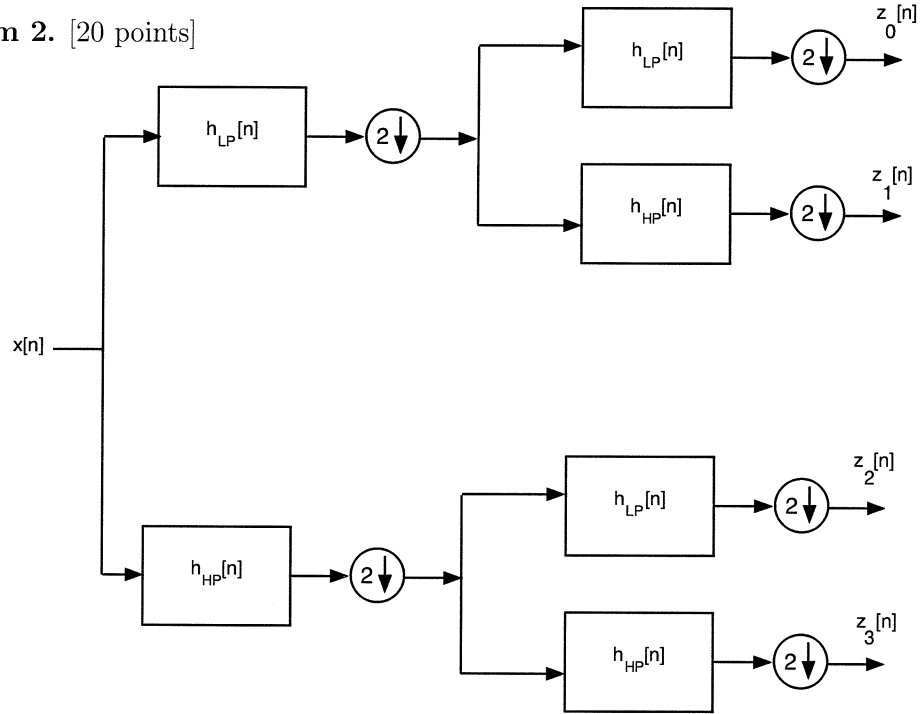


Figure 3(a). Analysis Section of Two-Stage Tree-Structured Filter Bank.

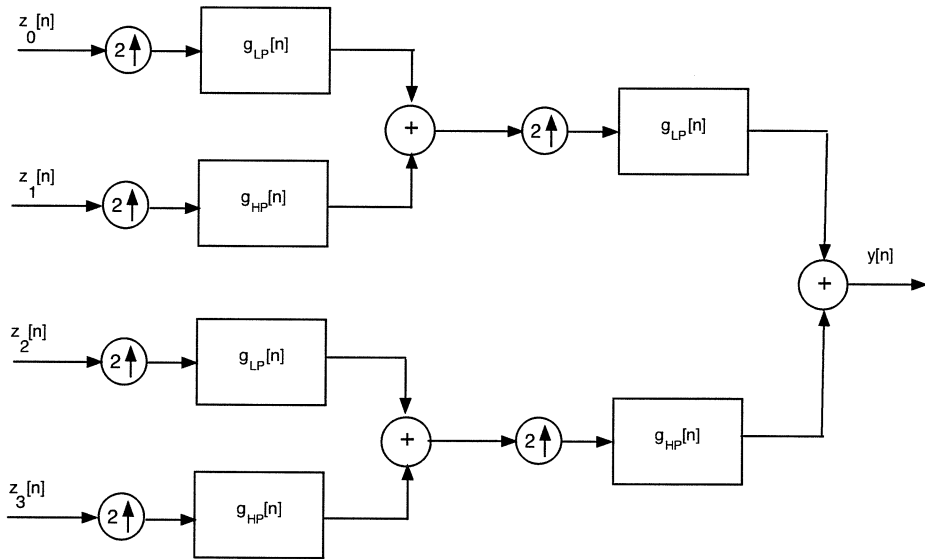


Figure 3(b). Synthesis Section of Two-Stage Tree-Structured Filter Bank.

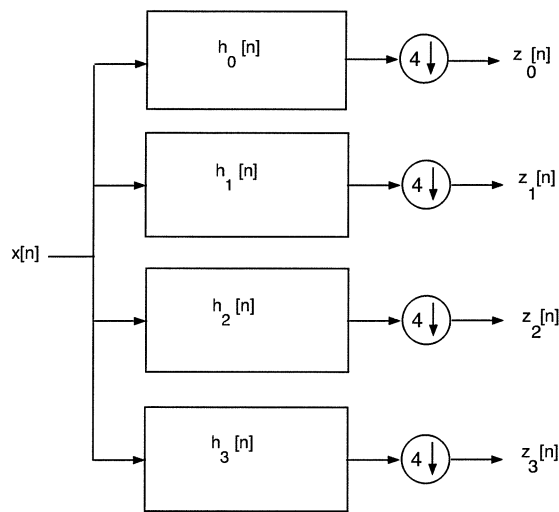


Figure 4(a). Analysis Filter Bank.

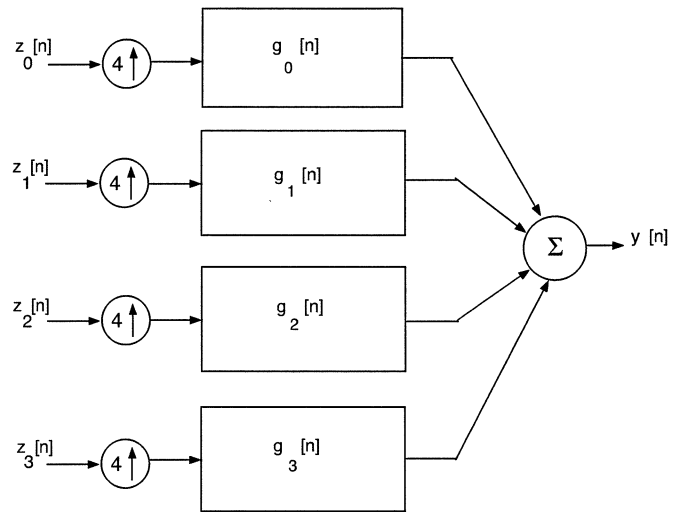


Figure 4(b). Synthesis Filter Bank.

This problem is about synthesizing the  $M=4$  channel uniform PR filter bank in Figure 4 from the two stage tree-structured PR filter bank in Figure 3.

Relative to Figure 3, the combination of the analysis filter pair,  $\{h_{LP}[n], h_{HP}[n]\}$ , and the synthesis filter pair  $\{g_{LP}[n], g_{HP}[n]\}$ , form a two-channel PR filter bank with

$$h_{HP}[n] = (-1)^n h_{LP}[n]$$

$$g_{LP}[n] = h_{LP}[n]$$

$$g_{HP}[n] = -h_{HP}[n]$$

- (a) For this part of the problem,  $h_{LP}[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$ , the simple two-tap filter studied in class and used in parts of Matlab Homework 2.

Use Noble's Downsampling Identity to determine the analysis filters  $h_k[n]$ ,  $k = 0, 1, 2, 3$ , so that the system in Fig. 4(a) is equivalent to the system in Fig. 3(a).

You need to show four answers on the next page. You can either write each  $h_k[n]$  in sequence form or do a stem plot. Show all work in arriving at your answers.

$$\begin{aligned}
 h_0[n] &= h_{LP}[n] * \sum_k h_{LP}[k] \delta[n-2k] \\
 &= \{1, 1\} * \{1, 0, 1\} = \{1, 1\} * \{\delta[n] + \delta[n-2]\} \\
 &= \{1, 1, 1, 1\} \\
 h_1[n] &= h_{LP}[n] * \sum_k h_{HP}[k] \delta[n-2k] \\
 &= \{1, 1\} * \{1, 0, -1\} = \{1, 1\} * \{\delta[n] - \delta[n-2]\} \\
 &= \{1, 1, -1, -1\}
 \end{aligned}$$

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$$h_2[n] = h_{HP}[n] * \sum_k h_{LP}[k] \delta[n-2k]$$

$$= \{1, -1\} * \{1, 0, 1\}$$

$$= \{1, -1, 1, -1\}$$

↑

$$h_3[n] = h_{HP}[n] * \sum_k h_{HP}[k] \delta[n-2k]$$

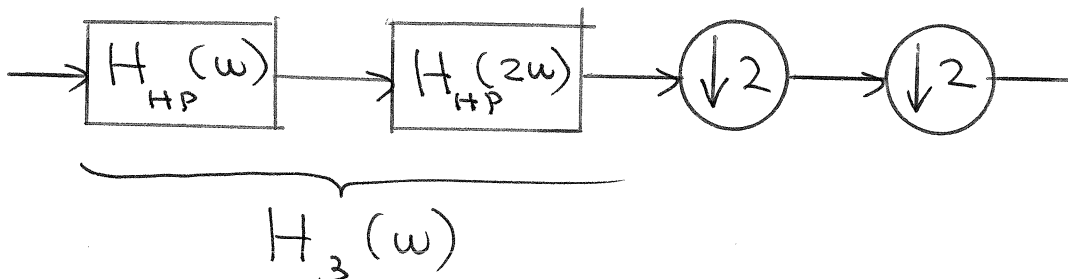
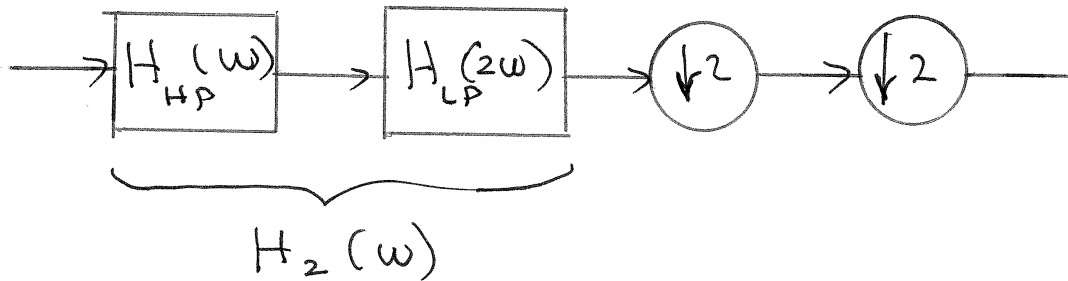
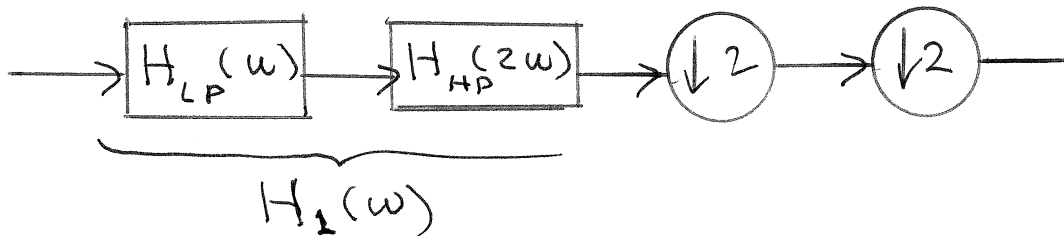
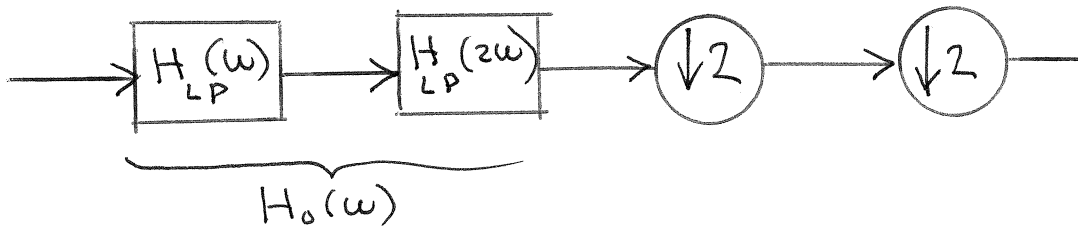
$$= \{1, -1\} * \{1, 0, -1\} = \{1, -1\} * \{\delta[n] - \delta[n-2]\}$$

$$= \{1, -1, -1, 1\}$$

↑

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## Applying Noble's Downsampling Identity



(b) For this part of the problem,  $h_{LP}[n]$  is now as defined below.

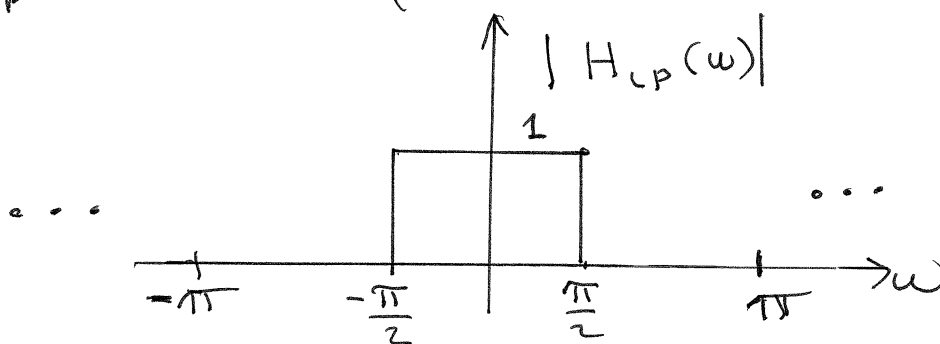
$$h_{LP}[n] = \frac{\sin\left(\frac{\pi}{2}(n+0.5)\right)}{\pi(n+0.5)}$$

You already used Noble's Downsampling Identity to convert the two-stage tree structure in Fig. 3(a) to the uniform filter bank in Fig. 4(a) for part (a) of this problem. All we did for this part of the problem is change  $h_{LP}[n]$  to have nicer plots in the frequency domain.

Plot the magnitude of the frequency response for each of the four analysis filters  $H_k(\omega)$ ,  $k = 0, 1, 2, 3$ . There should be four DTFT magnitude plots all together, each plotted over  $-\pi < \omega < \pi$ .

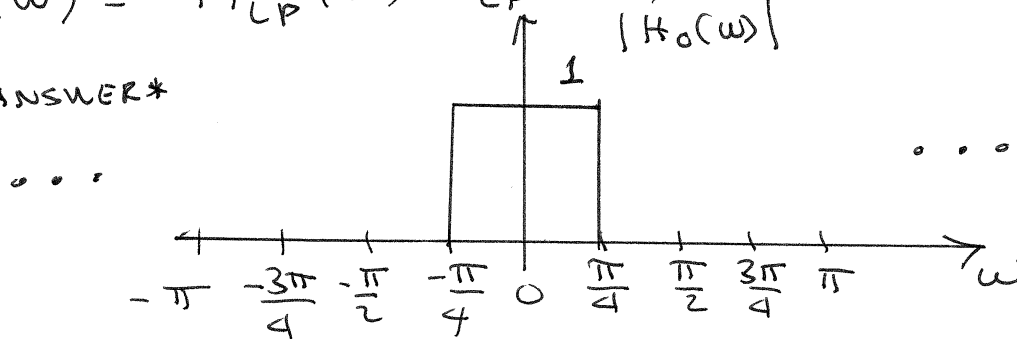
Note that the fractional time-shift by 0.5 in the definition of  $h_{LP}[n]$  above has no impact on the requested magnitude plots, so you can ignore it when you solve this part of the problem.

First, note: for  $|\omega| < \pi$ , the DTFT of  $h_{LP}[n]$  is

$$H_{LP}(\omega) = \text{rect}\left(\frac{\omega}{\pi}\right) e^{j\frac{\omega}{2}} \quad \text{for } -\pi < \omega < \pi$$


$$H_0(\omega) = H_{LP}(\omega) H_{LP}(2\omega)$$

\*ANSWER\*

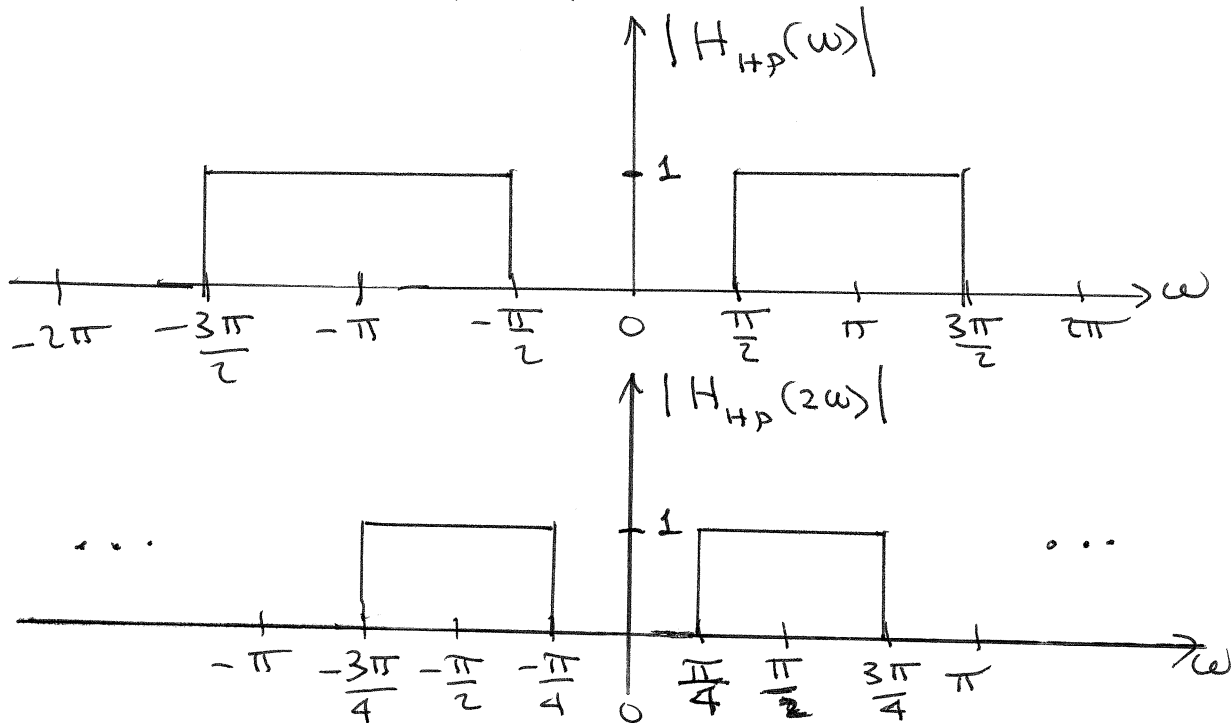


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of next  
page for  
 $H_{LP}(2\omega)$

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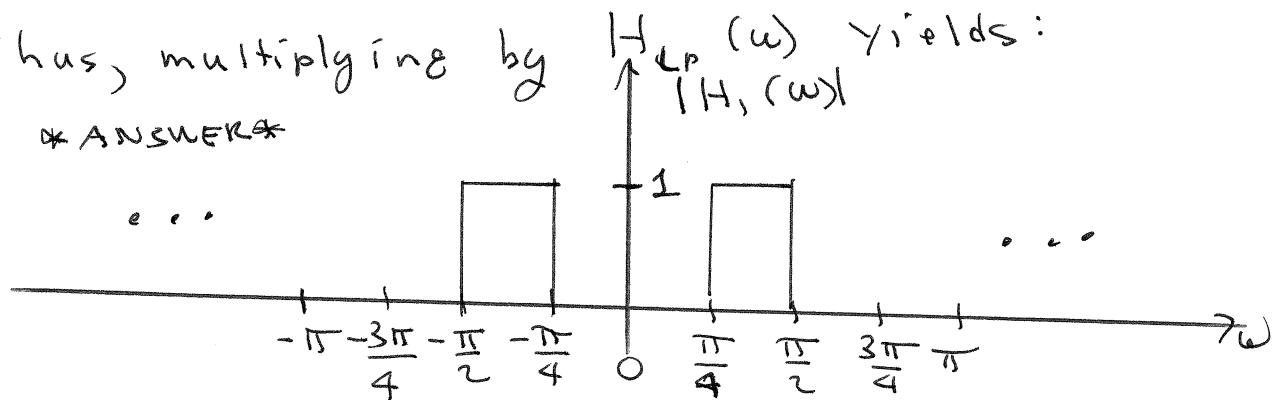
$$H_1(\omega) = H_{LP}(\omega) H_{HP}(2\omega)$$

What does  $|H_{HP}(2\omega)|$  do over  $-\pi < \omega < \pi$

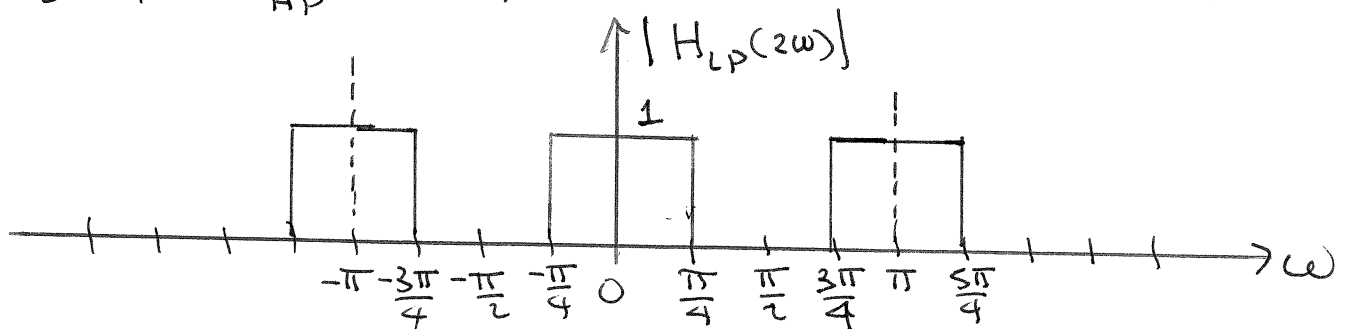


Thus, multiplying by  $H_{LP}(\omega)$  yields:

\*ANSWER\*

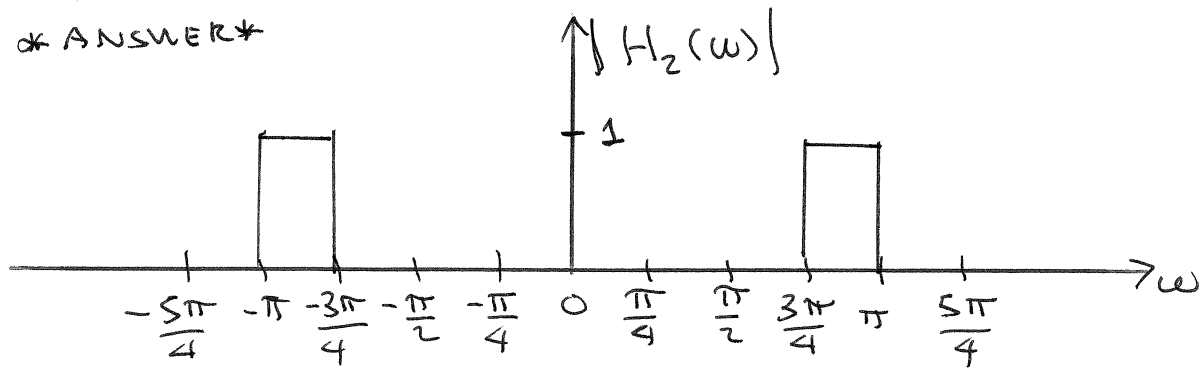


$$H_2(\omega) = H_{HP}(\omega) H_{LP}(2\omega) \Rightarrow \text{so let's look at } |H_{LP}(2\omega)|$$



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\* ANSWER \*



$$H_3(\omega) = H_{HP}(\omega) H_{HP}(2\omega)$$

\* ANSWER \*

