Digital Signal Processing I **Exam 3** **Fall 2006**
Session 26 **Live: 21 Nov. 2006**

Cover Sheet

Test Duration: 75 minutes.
Open Book but Closed Notes.
Calculators allowed.
This test contains **three** problems.
All work should be done in the blue books provided.
Do **not** return this test sheet, just return the blue books.

Digital Signal Processing I — Exam 3 — Fall 2006
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Problem 1. [40 pts]

(a) Let $X_9(k) = X(2\pi k/9)$, where $X(\omega)$ is the DTFT of the sequence

$$x[n] = (0.9)^n u[n] \overset{DTFT}{\longleftrightarrow} X(\omega) = \frac{1}{1 - 0.9e^{-j\omega}}$$

That is, $X_9(k)$ is the frequency domain sequence obtained by sampling $X(\omega)$ at $N = 9$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 9-pt inverse DFT of $X_9(k)$ may be expressed as

$$x_9[n] = \sum_{\ell=-\infty}^{\infty} x[n - \ell 9]\{u[n] - u[n-9]\} \underset{9}{\overset{DFT}{\longleftrightarrow}} X_9(k) = \frac{1}{1 - 0.9e^{-j2\pi k/9}}; k = 0, 1, ..., 8$$

Determine a simple, closed-form expression for $x_9[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers. *Hint:*

$$\frac{1}{1 - (.9)^9} = 1.6324$$

(b) Consider normalizing $x_9[n]$ so that it's first value is one, $\tilde{x}_9[n] = x_9[n]/x_9[0]$, $n = 0, 1, ..., 8$. Compare $\tilde{x}_9[n]$ and $x[n]$ over $n = 0, 1, ..., 8$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

(c) Let $Y_9(k) = Y(2\pi k/9)$, where $Y(\omega)$ is the DTFT of the sequence

$$y[n] = (0.9)^{|n-4|} \overset{DTFT}{\longleftrightarrow} Y(\omega) = e^{-j4\omega}\left\{\frac{1 - (.9)^2}{1 - 2(0.9)\cos(\omega) + (.9)^2}\right\}$$

That is, $Y_9(k)$ is the frequency domain sequence obtained by sampling $Y(\omega)$ at $N = 9$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 9-pt inverse DFT of $Y_9(k)$ may be expressed as

$$y_9[n] = \sum_{\ell=-\infty}^{\infty} y[n-\ell 9]\{u[n]-u[n-9]\} \underset{9}{\overset{DFT}{\longleftrightarrow}} Y_9(k) = \left\{\frac{1 - (.9)^2}{1 - 2(0.9)\cos(2\pi k/9) + (.9)^2}\right\} e^{-j8\pi k/9}$$

where $k = 0, 1, ..., 8$ for the right hand side directly above. Determine a simple, closed-form expression for $y_9[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers.

(d) Consider normalizing $y_9[n]$ so that it's first value is one, $\tilde{y}_9[n] = y_9[n]/y_9[0]$, $n = 0, 1, ..., 8$. Compare $\tilde{y}_9[n]$ and $y[n]$ over $n = 0, 1, ..., 8$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

Problem 2. [30 points]

Consider an input sequence, $x[n]$, of length $L = 6$ and an FIR filter with impulse response $h[n]$ of length $M = 6$ as described below.

$$\begin{aligned} x[n] &= u[n] - u[n-6] = \{1, 1, 1, 1, 1, 1\} \\ h[n] &= u[n] - u[n-6] = \{1, 1, 1, 1, 1, 1\} \end{aligned}$$

We compute an $N = 8$-pt. DFT of each of these two sequences as

$$x[n] \underset{8}{\overset{\text{DFT}}{\longleftrightarrow}} X_8[k] \qquad\qquad h[n] \underset{8}{\overset{\text{DFT}}{\longleftrightarrow}} H_8[k]$$

Next, we point-wise multiply the DFT sequences to form $Y_8[k] = X_8[k]H_8[k]$, $k = 0, 1, ..., 7$.. Finally, we compute an $N = 8$-pt. inverse DFT of $Y_8[k]$ to obtain $y_P[n]$. Determine the numerical values of $y_P[n]$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$..

Problem 3. [30 points]

As part of the first stage in a radix 2 FFT, a sequence $x[n]$ of length $N = 8$ is decomposed into two sequences of length 4 as

$$f_0[n] = x[2n], \quad n = 0, 1, 2, 3 \qquad\qquad f_1[n] = x[2n+1], \quad n = 0, 1, 2, 3$$

We compute a 4 -pt. DFT of each of these two sequences as

$$f_0[n] \underset{4}{\overset{\text{DFT}}{\longleftrightarrow}} F_0[k] \qquad\qquad f_1[n] \underset{4}{\overset{\text{DFT}}{\longleftrightarrow}} F_1[k]$$

The specific values of $F_0[k]$ and $F_1[k]$, $k = 0, 1, 2, 3$, obtained from the length $N = 8$ sequence in question are listed in the Table below.

k	0	1	2	3
$F_0[k]$	0	4	4	0
$F_1[k]$	0	$\sqrt{2}(2+2j)$	$4j$	0
W_8^k	1	$\frac{1}{\sqrt{2}}(1-j)$	-j	$-\frac{1}{\sqrt{2}}(1+j)$

(20 pts) From the values of $F_0[k]$ and $F_1[k]$, $k = 0, 1, 2, 3$, and the values of $W_8^k = e^{-j\frac{2\pi}{8}k}$, $k = 0, 1, 2, 3$, provided in the Table, determine the numerical values of the actual $N = 8$-pt. DFT of $x[n]$ denoted $X_8[k]$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$.

(10 pts) The underlying length $N = 8$ sequence $x[n]$ may be expressed as

$$x[n] = e^{j2\pi\frac{k_1}{8}n} + e^{j2\pi\frac{k_2}{8}n}, \quad n = 0, 1, ..., 7.$$

where k_1 and k_2 are both integers between 0 and 7, that is, $k_i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$, $i = 1, 2$. Given the values of $X_8[k]$ for $k = 0, 1, ..., 7$ determined in part (a), determine the numerical values of k_1 and k_2.