Cover Sheet

Test Duration: 75 minutes.
Open Book but Closed Notes.
Calculators allowed.
This test contains three problems.
All work should be done in the blue books provided.
Do not return this test sheet, just return the blue books.
Problem 1. [40 pts]

(a) Let $X_9(k) = X(2\pi k/9)$, where $X(\omega)$ is the DTFT of the sequence

$$x[n] = (0.9)^n u[n] \quad \text{DTFT} \quad \longleftrightarrow \quad X(\omega) = \frac{1}{1 - 0.9e^{-j\omega}}$$

That is, $X_9(k)$ is the frequency domain sequence obtained by sampling $X(\omega)$ at $N = 9$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 9-pt inverse DFT of $X_9(k)$ may be expressed as

$$x_9[n] = \sum_{\ell=\infty}^{\infty} x[n-\ell 9] \{u[n] - u[n-9]\} \quad \text{DFT} \quad \longleftrightarrow \quad X_9(k) = \frac{1}{1 - 0.9e^{-j2\pi k/9}}, k = 0, 1, ..., 8$$

Determine a simple, closed-form expression for $x_9[n]$. A closed-form expression contains NO summations and it is NOT a listing of numbers. Hint:

$$\frac{1}{1 - (0.9)^9} = 1.6324$$

(b) Consider normalizing $x_9[n]$ so that it’s first value is one, $\tilde{x}_9[n] = x_9[n]/x_9[0]$, $n = 0, 1, ..., 8$. Compare $\tilde{x}_9[n]$ and $x[n]$ over $n = 0, 1, ..., 8$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

(c) Let $Y_9(k) = Y(2\pi k/9)$, where $Y(\omega)$ is the DTFT of the sequence

$$y[n] = (0.9)^{n-4} \quad \text{DTFT} \quad \longleftrightarrow \quad Y(\omega) = e^{-j4\omega} \left\{ \frac{1 - (0.9)^2}{1 - 2(0.9)\cos(\omega) + (0.9)^2} \right\}$$

That is, $Y_9(k)$ is the frequency domain sequence obtained by sampling $Y(\omega)$ at $N = 9$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 9-pt inverse DFT of $Y_9(k)$ may be expressed as

$$y_9[n] = \sum_{\ell=\infty}^{\infty} y[n-\ell 9] \{u[n] - u[n-9]\} \quad \text{DFT} \quad \longleftrightarrow \quad Y_9(k) = \left\{ \frac{1 - (0.9)^2}{1 - 2(0.9)\cos(2\pi k/9) + (0.9)^2} \right\} e^{-j8\pi k/9}$$

where $k = 0, 1, ..., 8$ for the right hand side directly above. Determine a simple, closed-form expression for $y_9[n]$. A closed-form expression contains NO summations and it is NOT a listing of numbers.

(d) Consider normalizing $y_9[n]$ so that it’s first value is one, $\tilde{y}_9[n] = y_9[n]/y_9[0]$, $n = 0, 1, ..., 8$. Compare $\tilde{y}_9[n]$ and $y[n]$ over $n = 0, 1, ..., 8$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.
Problem 2. [30 points]

Consider an input sequence, \( x[n] \), of length \( L = 6 \) and an FIR filter with impulse response \( h[n] \) of length \( M = 6 \) as described below.

\[
\begin{align*}
x[n] &= u[n] - u[n-6] = \{1, 1, 1, 1, 1, 1\} \\
h[n] &= u[n] - u[n-6] = \{1, 1, 1, 1, 1, 1\}
\end{align*}
\]

We compute an \( N = 8 \)-pt. DFT of each of these two sequences as

\[
\begin{align*}
x[n] &\longleftrightarrow X_8[k] \\
h[n] &\longleftrightarrow H_8[k]
\end{align*}
\]

Next, we point-wise multiply the DFT sequences to form \( Y_8[k] = X_8[k]H_8[k] \), \( k = 0, 1, ..., 7 \).

Finally, we compute an \( N = 8 \)-pt. inverse DFT of \( Y_8[k] \) to obtain \( y_P[n] \). Determine the numerical values of \( y_P[n] \) for \( n = 0, 1, 2, 3, 4, 5, 6, 7 \).

Problem 3. [30 points]

As part of the first stage in a radix 2 FFT, a sequence \( x[n] \) of length \( N = 8 \) is decomposed into two sequences of length 4 as

\[
\begin{align*}
f_0[n] &= x[2n], \ n = 0, 1, 2, 3 \\
f_1[n] &= x[2n + 1], \ n = 0, 1, 2, 3
\end{align*}
\]

We compute a 4-pt. DFT of each of these two sequences as

\[
\begin{align*}
f_0[n] &\longleftrightarrow F_0[k] \\
f_1[n] &\longleftrightarrow F_1[k]
\end{align*}
\]

The specific values of \( F_0[k] \) and \( F_1[k] \), \( k = 0, 1, 2, 3 \), obtained from the length \( N = 8 \) sequence in question are listed in the Table below.

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_0[k] )</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>( F_1[k] )</td>
<td>0</td>
<td>( \sqrt{2}(2 + 2j) )</td>
<td>4j</td>
<td>0</td>
</tr>
<tr>
<td>( W_8^k )</td>
<td>1</td>
<td>( \frac{1}{\sqrt{2}}(1 - j) )</td>
<td>-j</td>
<td>( -\frac{1}{\sqrt{2}}(1 + j) )</td>
</tr>
</tbody>
</table>

(20 pts) From the values of \( F_0[k] \) and \( F_1[k] \), \( k = 0, 1, 2, 3 \), and the values of \( W_8^k = e^{-j\frac{2\pi k}{8}} \), \( k = 0, 1, 2, 3 \), provided in the Table, determine the numerical values of the actual \( N = 8 \)-pt. DFT of \( x[n] \) denoted \( X_8[k] \) for \( k = 0, 1, 2, 3, 4, 5, 6, 7 \).

(10 pts) The underlying length \( N = 8 \) sequence \( x[n] \) may be expressed as

\[
x[n] = e^{j\frac{2\pi k_1 n}{N}} + e^{j\frac{2\pi k_2 n}{N}}, \ n = 0, 1, ..., 7.
\]

where \( k_1 \) and \( k_2 \) are both integers between 0 and 7, that is, \( k_i \in \{0, 1, 2, 3, 4, 5, 6, 7\} \), \( i = 1, 2 \). Given the values of \( X_8[k] \) for \( k = 0, 1, ..., 7 \) determined in part (a), determine the numerical values of \( k_1 \) and \( k_2 \).