EE538 **Exam 3** **Fall 2002**
Digital Signal Processing I **Nov. 24, 2003**

Cover Sheet

Test Duration: 50 minutes.
Open Book but Closed Notes.
Calculators **not** allowed
This test contains **three** problems.
All work should be done in the blue books provided.
You must show all work for each problem to receive full credit.
Do **not** return this test sheet, just return the blue books.

No.	**Topic(s) of Problem**	**Points**
1.	Basics of AR spectral estimation; relationship between AR and linear prediction	30
2.	Sampling of DTFT and Time-Domain Aliasing	30
3.	Autoregressive spectral estimation; basics of AR, MA, and ARMA processes.	40

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Problem 1. [30 points]

Consider the autoregressive AR(3) process generated via the difference equation

$$x[n] = \frac{1}{2}x[n-1] - \frac{1}{4}x[n-2] + \frac{1}{8}x[n-3] + \nu[n]$$

where $\nu[n]$ is a stationary, white noise process with variance $\sigma_w^2 = 2$.

(a) Determine a simple closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m] = E\{x[n]x[n-m]\}$:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]e^{-jm\omega}$$

(b Consider the third-order predictor

$$\hat{x}[n] = -a_3(1)x[n-1] - a_3(2)x[n-2] - a_3(3)x[n-3]$$

Determine the numerical values of the optimum predictor coefficients $a_3(1)$, $a_3(2)$, and $a_3(3)$ **and** compute the corresponding minimum mean-square error.

(b Consider the fourth-order predictor

$$\hat{x}[n] = -a_4(1)x[n-1] - a_4(2)x[n-2] - a_4(3)x[n-3] - a_4(4)x[n-4]$$

Determine the numerical values of the optimum predictor coefficients $a_4(1)$, $a_4(2)$, $a_4(3)$, and $a_4(4)$ **and** compute the corresponding minimum mean-square error.

Problem 2. [30 points]

(a) Let $X_8(k) = X(2\pi k/8)$, where $X(\omega)$ is the DTFT of the sequence

$$x[n] = (0.9)^n e^{j\frac{\pi}{4}n}u[n] \overset{DTFT}{\longleftrightarrow} X(\omega) = \frac{1}{1 - 0.9e^{j\frac{\pi}{4}}e^{-j\omega}}$$

That is, $X_8(k)$ is what we obtain by sampling $X(\omega)$ at $N = 8$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 8-pt inverse DFT of $X_8(k)$ may be expressed as

$$x_8[n] = \sum_{\ell=-\infty}^{\infty} x[n-\ell 8]\{u[n]-u[n-8]\} \underset{8}{\overset{DFT}{\longleftrightarrow}} X_8(k) = \frac{1}{1 - 0.9e^{j\frac{\pi}{4}}e^{-j2\pi k/8}}; k = 0, 1, ..., 7$$

Determine a simple, closed-form expression for $x_8[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers. *Hint:*

$$\frac{1}{1-(.9)^8} = 1.7558$$

(b) Consider normalizing $x_8[n]$ so that it's first value is one, $\tilde{x}_8[n] = x_8[n]/x_8[0]$, $n = 0, 1, ..., 7$. Compare $\tilde{x}_8[n]$ and $x[n]$ over $n = 0, 1, ..., 7$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

(c) Let $Y_8(k) = Y(2\pi k/8)$, where $Y(\omega)$ is the DTFT of the sequence

$$y[n] = 2\ (0.9)^n cos\left(\frac{\pi}{4}\ n\right) u[n]$$

That is, $Y_8(k)$ is what we obtain by sampling $Y(\omega)$ at $N = 8$ equi-spaced points in the interval $0 \leq \omega < 2\pi$. Theory derived in class and in the textbook dictates that the 8-pt inverse DFT of $Y_8(k)$ may be expressed as

$$y_8[n] = \sum_{\ell=-\infty}^{\infty} y[n-\ell 8]\{u[n]-u[n-8]\} \overset{DFT}{\underset{8}{\longleftrightarrow}} Y_8(k) = \frac{1}{1-0.9e^{j\frac{\pi}{4}}e^{-j2\pi k/8}} + \frac{1}{1-0.9e^{-j\frac{\pi}{4}}e^{-j2\pi k/8}}$$

Determine a simple, closed-form expression for $y_8[n]$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers.

(d) Consider normalizing $y_8[n]$ so that it's first value is one, $\tilde{y}_8[n] = y_8[n]/y_8[0]$, $n = 0, 1, ..., 7$. Compare $\tilde{y}_8[n]$ and $y[n]$ over $n = 0, 1, ..., 7$. Are they the same or different? Briefly explain your answer as to why or why not they are the same.

Problem 3. [40 points]

Consider the ARMA(1,1) process generated via the difference equation

$$x[n] = \frac{1}{2}x[n-1] + \nu[n] - 2\nu[n-1]$$

where $\nu[n]$ is a stationary white noise process with variance $\sigma_w^2 = 1$.

(a) Determine a closed-form expression for the autocorrelation sequence $r_{xx}[m] = E\{x[n]x[n-m]\}$ which holds for $-\infty < m < \infty$. A *closed-form* expression contains NO summations and it is NOT a listing of numbers.

(b) Determine a **simple** closed-form expression for the spectral density for $x[n]$, $S_{xx}(\omega)$, which may be expressed as the DTFT of $r_{xx}[m] = E\{x[n]x[n-m]\}$. Note, though, that there is a way to determine $S_{xx}(\omega)$ without computing the DTFT of $r_{xx}[m]$. You will not receive much credit if you don't simplify as much as possible. Plot $S_{xx}(\omega)$ for $-\pi < \omega < \pi$.

(c) Consider that the power spectrum of the ARMA(1,1) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}^{(1)}}{|1 + a_1^{(1)}e^{-j\omega}|^2}$$

Determine the respective numerical values of the optimum first-order linear prediction coefficient $a_1^{(1)}$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(1)}$.

(d) Consider that the power spectrum of the ARMA(1,1) process $x[n]$ is estimated via AR spectral estimation according to

$$S_{xx}(\omega) = \frac{\mathcal{E}_{min}^{(2)}}{|1 + a_1^{(2)} e^{-j\omega} + a_2^{(2)} e^{-j2\omega}|^2}$$

Determine the respective numerical values of the optimum second-order linear prediction coefficients $a_1^{(2)}$ and $a_2^{(2)}$ and the value of the corresponding minimum mean-square error $\mathcal{E}_{min}^{(2)}$.