EE 538 Exam 3 Sol'ns. Fall 2001

Prob. 1 $r_{xx}[m] = -a_1 r_{xx}[m-1] - a_2 r_{xx}[m-2] + \sigma_w^2 \delta[m]$

(a)

$$r_{xx}[m] = +\frac{7}{12} r_{xx}[m-1] - \frac{2}{12} r_{xx}[m-2] + \frac{35}{6}\delta[m]$$

· Set 3 eqns. in 3 unknowns: $\boxed{\begin{array}{l} r_{xx}[-1] = r_{xx}[1] \\ r_{xx}[-2] = r_{xx}[2] \end{array}}$

$$\begin{array}{c} m=0 \\ m=1 \\ m=2 \end{array} \begin{bmatrix} 1 & \frac{-7}{12} & \frac{2}{12} \\ \frac{-7}{12} & 1+\frac{2}{12} & 0 \\ \frac{2}{12} & -\frac{7}{12} & 1 \end{bmatrix} \begin{bmatrix} r_{xx}[0] \\ r_{xx}[1] \\ r_{xx}[2] \end{bmatrix} = \begin{bmatrix} 35/6 \\ 0 \\ 0 \end{bmatrix}$$

· multiply by 12 on both sides:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 12 & -7 & 2 \\ -7 & 14 & 0 \\ 2 & -7 & 12 \end{bmatrix} \begin{bmatrix} r_{xx}[0] \\ r_{xx}[1] \\ r_{xx}[2] \end{bmatrix} = \begin{bmatrix} 70 \\ 0 \\ 0 \end{bmatrix}$$

$r_{xx}[0] = 8$

$r_{xx}[1] = 4$

$r_{xx}[2] = 1$

(b)

$$S_{xx}(\omega) = \frac{35/6}{\left| 1 - \frac{7}{12} e^{-j\omega} + \frac{1}{6} e^{-j2\omega} \right|^2}$$

$\mathcal{E}_1 = r_{xx}[0]\{1 - a_1^2(1)\}$

$= 8\{1 - (\frac{1}{4})\} = 6$

(c) $a_1(1) = -r_{xx}[1]/r_{xx}[0] = -4/8 = -1/2$

(d) $a_3(1) = -7/12$, $a_3(2) = \frac{2}{12}$, $a_3(3) = 0$

$\mathcal{E}_3 = \sigma_w^2 = 35/6$

Problem 2 $\quad r_{xx}[m] = -a_1 r_{xx}[m-1] - a_2 r_{xx}[m-2]$

$$\begin{array}{c} m=1 \\ m=2 \end{array} \begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2-j \\ -2+j & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2-j \\ -1 \end{bmatrix}$$

$$a_1 = 1-j \qquad a_2 = -j$$

$$z^2 + a_1 z + a_2$$

$$= z^2 + \underbrace{(1-j)}_{\text{sum of roots}} z \underbrace{- j}_{\text{product of roots}}$$

$$= (z+1)(z-j)$$

$$-1 = e^{j\pi} \Rightarrow \omega_1 = \pi$$

$$j = e^{j\frac{\pi}{2}} \Rightarrow \omega_2 = \frac{\pi}{2}$$

(b) Sum of p complex sinewaves perfectly predicted from p past values

$$a_1(1) = a_1 = 1-j \qquad a_2(2) = a_2 = -j$$

Problem 3

$$\frac{X(z)}{W(z)} = H(z) = \frac{z+1}{z-\frac{1}{2}}$$

$$r_{xx}[m] = r_{hh}[m] \cdot \underbrace{\sigma_w^2}_{=1} = r_{hh}[m]$$

$$= h[m] * h[-m]$$

$h[n] = ?$

$$z - \tfrac{1}{2}\ \overline{\big)\ z + 1\ }^{\,1}$$
$$\underline{-(z - \tfrac{1}{2})}$$
$$\tfrac{3}{2}$$

$$\frac{z+1}{z-\frac{1}{2}} = 1 + \frac{3}{2}\,\frac{1}{z-\frac{1}{2}} = 1 + \frac{3}{2} z^{-1} \frac{z}{z-\frac{1}{2}}$$

$$h[n] = \delta[n] + \frac{3}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= 3\left(\frac{1}{2}\right)^n u[n] - 2\delta[n]$$

$$= \left\{ 3\left(\frac{1}{2}\right)^n u[n] - 2\delta[n] \right\}$$

$$r_{hh}[m] = h[m] * h[-m]$$

$$= \left\{ 3\left(\frac{1}{2}\right)^m u[m] - 2\delta[m] \right\} * \left\{ 3\left(\frac{1}{2}\right)^{-m} u[-m] - 2\delta[-m] \right\}$$

$$= \left\{ 9 \underbrace{\frac{1}{1-\left(\frac{1}{2}\right)^2}}_{\frac{4}{3}} \cdot \left(\frac{1}{2}\right)^{|m|} - 6\left(\frac{1}{2}\right)^m u[m] - 6\left(\frac{1}{2}\right)^{-m} u[-m] + 4\delta[m] \right\}$$

Problem 3 (cont.)

$m=0$: $r_{hh}[0] = \left\{12 - 6 - 6 + 4\right\} = 4$

$m=1$: $r_{hh}[1] = \left\{12\left(\frac{1}{2}\right) - 6 \cdot \frac{1}{2}\right\} = 3$

$m=2$: $r_{hh}[2] = \left\{12\left(\frac{1}{4}\right) - 6\,\frac{1}{4}\right\} = 3 - \frac{3}{2} = \frac{3}{2}$

(b) $$S_{xx}(\omega) = \underbrace{(1)}_{\sigma_n^2} \frac{\left|1 + e^{-j\omega}\right|^2}{\left|1 - \frac{1}{2}e^{j\omega}\right|^2}$$

(c) $$a_1(1) = \frac{-r_{xx}[1]}{r_{xx}[0]} = \frac{-3}{4}$$

$$\mathcal{E}_1 = r_{xx}[0]\left\{1 - a_1^2(1)\right\} = 7/4$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = -\begin{bmatrix} 3 \\ 3/2 \end{bmatrix}$$

$a_1(1) = \frac{-15}{14}$ $\qquad a_2(2) = \frac{3}{7}$

$$\mathcal{E}_2 = \mathcal{E}_1\left(1 - \left(\frac{3}{7}\right)^2\right) = \frac{10}{7}$$