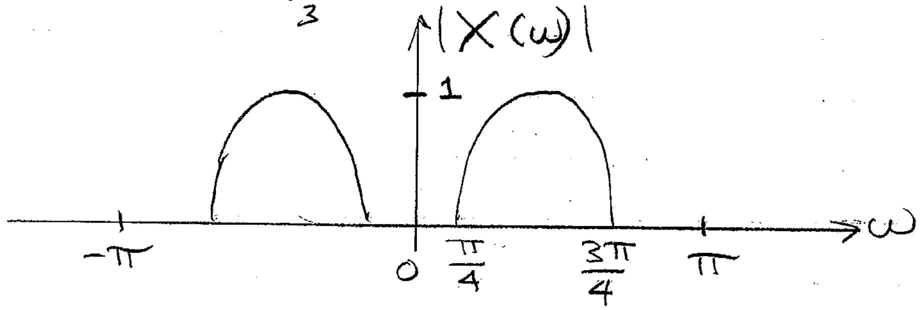
ECE 538   Solutions to Exam 2   1

F'05

Sol'n to Prob. 1

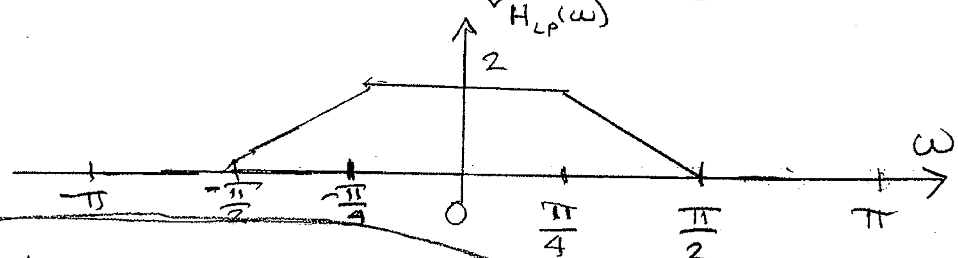
(a) $\omega_1 = 2\pi \dfrac{\frac{W}{3}}{8\frac{W}{3}} = \dfrac{\pi}{4}$

$\omega_2 = 2\pi \dfrac{W}{\frac{8}{3}W} = \dfrac{3\pi}{4}$



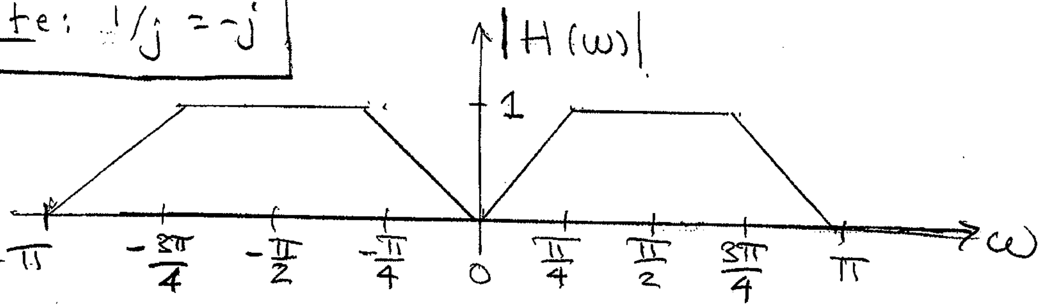
$\angle X(\omega) = 0 \quad \forall\, \omega$

(b) $h[n] = \underbrace{16\left\{\dfrac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \cdot \dfrac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\right\}}_{H_{LP}(\omega)} \sin\left(\frac{\pi}{2}n\right)$

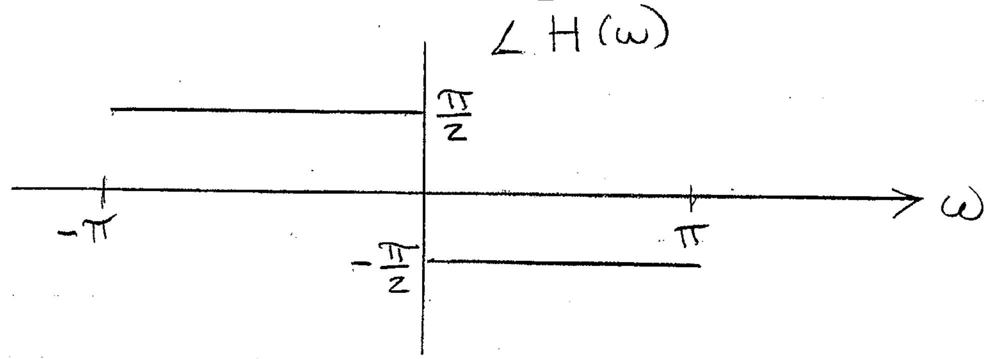


$H(\omega) = \dfrac{1}{2j} H_{LP}\left(\omega - \frac{\pi}{2}\right) - \dfrac{1}{2j} H_{LP}\left(\omega + \frac{\pi}{2}\right)$

$\boxed{\text{note: } 1/j = -j}$



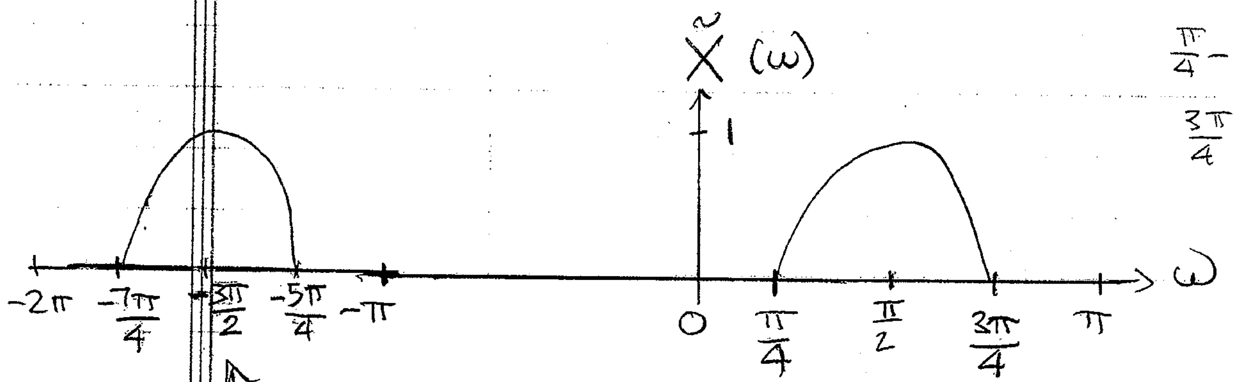
Sol'n to Prob. 1 (cont.)



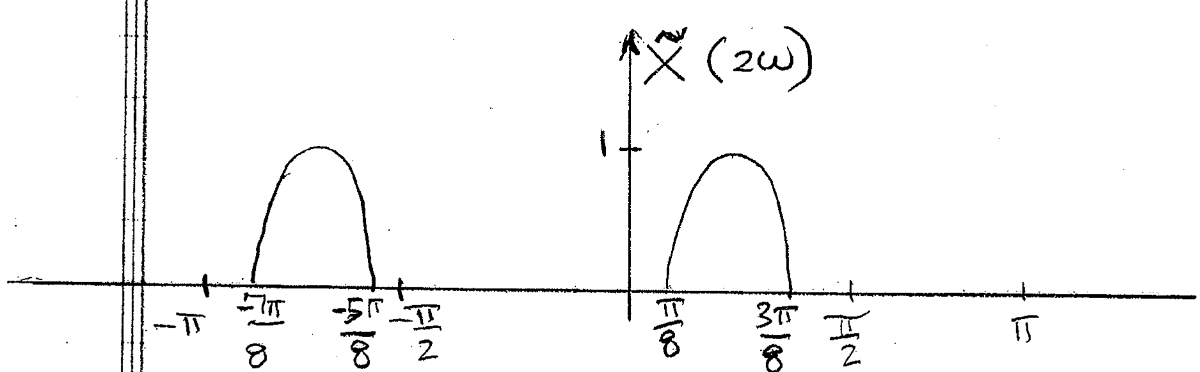
(c) Since $H(\omega) = -j$ over $\frac{\pi}{4} < \omega < \frac{3\pi}{4}$ and $+j$ for $-\frac{3\pi}{4} < \omega < -\frac{\pi}{4}$, $\hat{x}[n]$ is the Hilbert Transform of $x[n]$

$\frac{\pi}{4} - \frac{8\pi}{4}$

$\frac{3\pi}{4} - \frac{8\pi}{4}$

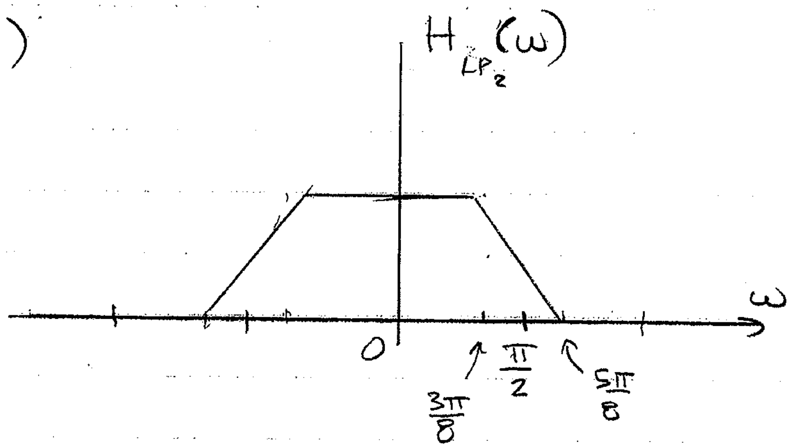


a DTFT is always periodic with period 2π



$\angle \tilde{X}(\omega) = 0$

(d)

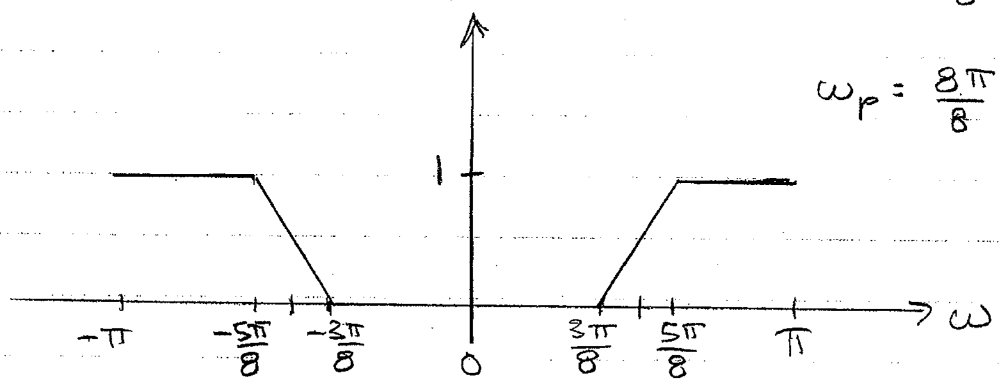


$$\frac{4\pi}{8} - \frac{\pi}{8} = \frac{3\pi}{8}$$

$$\frac{4\pi}{8} + \frac{\pi}{8} = \frac{5\pi}{8}$$

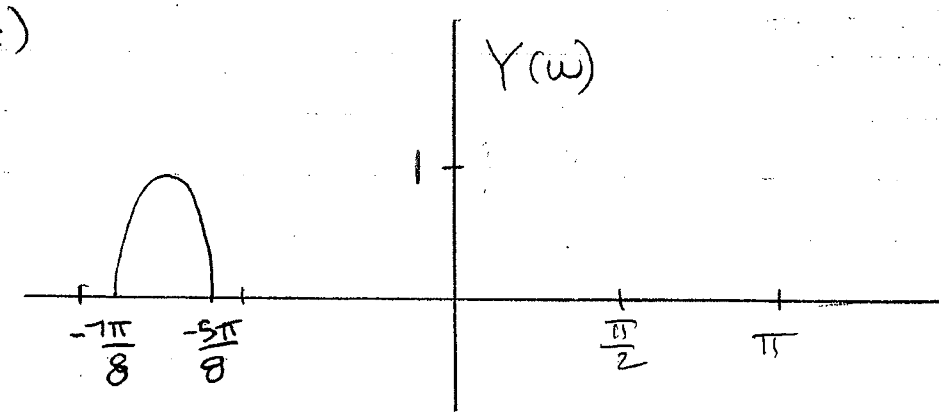
$$H_2(\omega) = H_{LP_2}(\omega - \pi) \Rightarrow \omega_s = \frac{8\pi}{8} - \frac{5\pi}{8} = \frac{3\pi}{8}$$

$$\omega_p = \frac{8\pi}{8} - \frac{3\pi}{8} = \frac{5\pi}{8}$$



$\angle H_2(\omega) = 0 \quad \forall \omega$

(e)



$\angle Y(\omega) = 0 \quad \forall \omega$

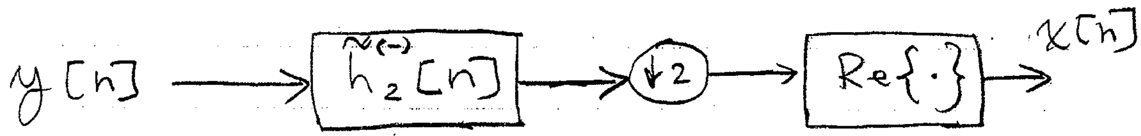
Prob. 1 (cont.)

(f)

$$y[n] = \tilde{x}_a\left(\frac{n}{2F_s}\right) e^{j\pi n}$$

$$y[2n] = \tilde{x}_a\left(\frac{2n}{2F_s}\right) e^{j2\pi n}$$

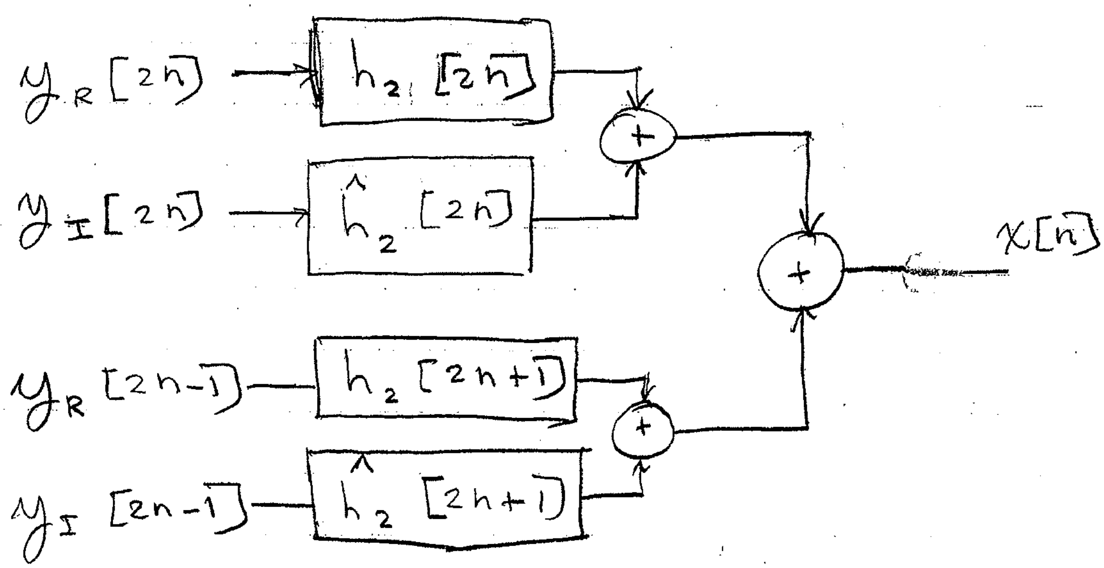
$$= \tilde{x}[n] = x[n] + j\,\hat{x}[n]$$



where: $\tilde{h}_2^{(-)} = h_2[n] - j\,\hat{h}_2[n]$

assume signals outside band occupied by $y[n]$

(g)



Sol'n to Prob. 2

Work in freq. domain:

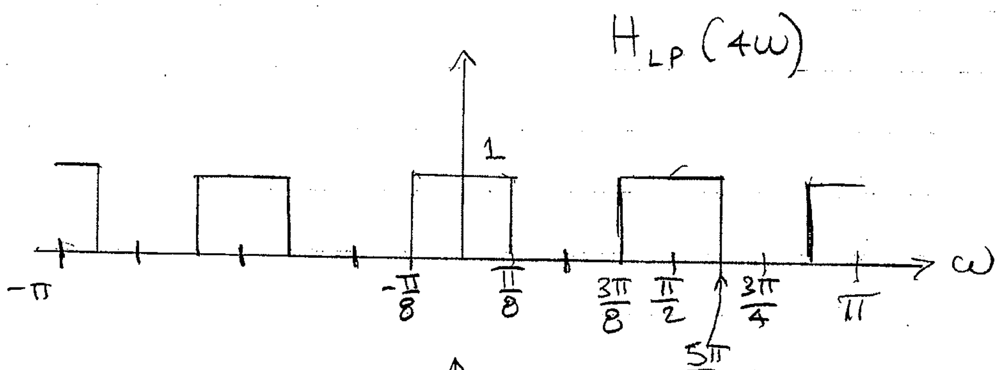
$$\to \boxed{\uparrow 2} \to \boxed{H_{LP}(\omega)} \to \boxed{2\uparrow} \to \boxed{H_{HP}(\omega)} \to \boxed{2\uparrow} \to \boxed{H_{LP}(\omega)} \to$$

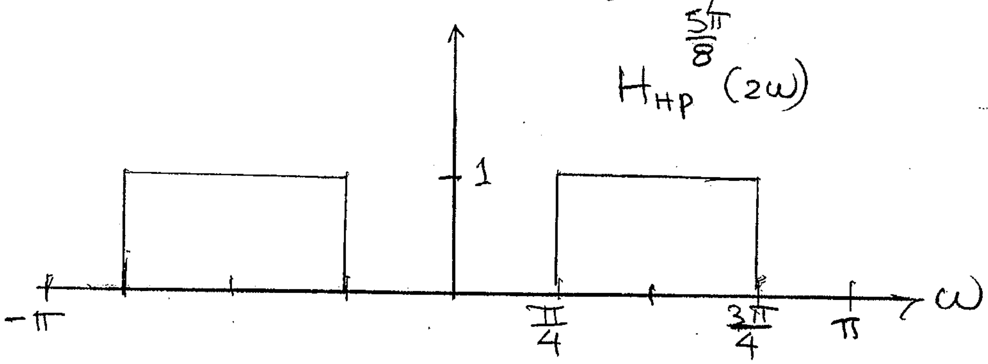
$$\to \boxed{2\uparrow} \to \boxed{2\uparrow} \to \boxed{H_{LP}(2\omega)} \to \boxed{H_{HP}(\omega)} \to \boxed{2\uparrow} \to \boxed{H_{LP}(\omega)} \to$$

$$\to \boxed{4\uparrow} \to \boxed{H_{LP}(2\omega)\, H_{HP}(\omega)} \to \boxed{2\uparrow} \to \boxed{H_{LP}(\omega)} \to$$

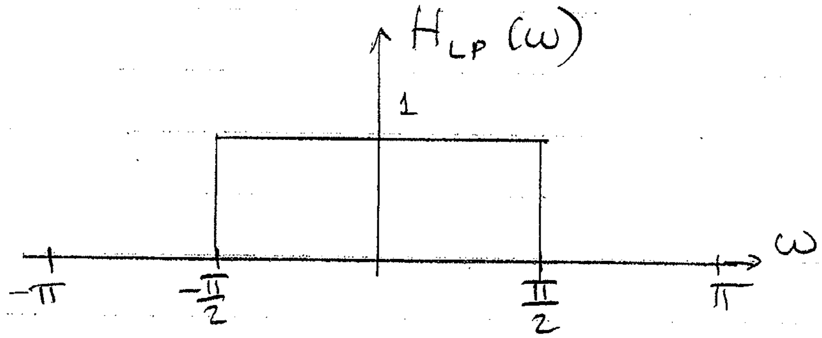
$$\to \boxed{4\uparrow} \to \boxed{2\uparrow} \to \boxed{H_{LP}(4\omega)\, H_{HP}(2\omega)} \to \boxed{H_{LP}(\omega)} \to$$

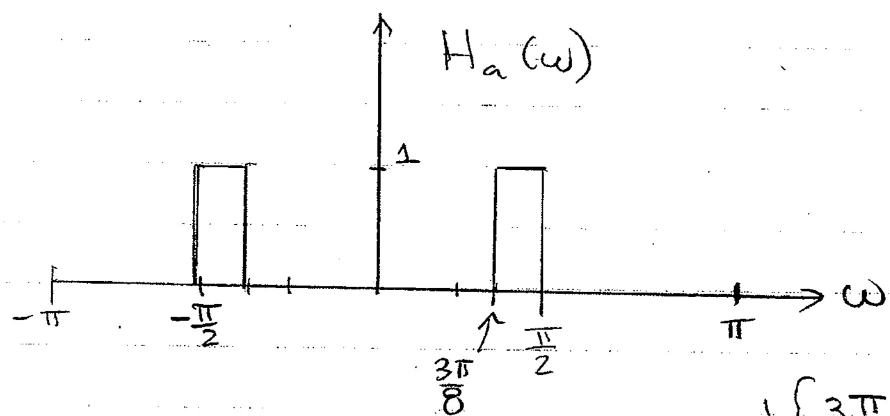
$$\to \boxed{8\uparrow} \to \boxed{H_{LP}(4\omega)\, H_{HP}(2\omega)\, H_{LP}(\omega)} \to$$





Sol'n to Prob. 2 (cont.)





$$\frac{1}{2}\left\{\frac{3\pi}{8} + \frac{4\pi}{8}\right\} = \frac{7\pi}{16}$$

$$h_a[n] = 2\left\{\frac{\sin\left(\frac{\pi}{16}n\right)}{\pi n}\right\}\cos\left(\frac{7\pi}{16}\right)$$

$$\frac{1}{2}\left\{\frac{4\pi}{8} - \frac{3\pi}{8}\right\} = \frac{\pi}{16}$$

Sol'n to Prob. 3 F '05 7

$$z = \frac{1+s}{1-s} \quad \text{and} \quad \omega = 2\tan^{-1}(\Omega)$$
(in textbook)

(a) Analog Filter has poles in LHP $\Rightarrow$ stable
bilinear transform maps stable analog filter to stable digital filter

(b) Zero of $H(s)$ at $s = j\sqrt{3}$ $\Rightarrow$ on imaginary axis $\Rightarrow$ $H(\Omega)$ at $\Omega = \sqrt{3}$ is zero

Guaranteed that "point" on $s = j\Omega$ axis is mapped to point on unit circle

$$\omega = 2\tan^{-1}(\sqrt{3}) \quad \Rightarrow \text{given} \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

THUS:
$$\omega = 2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3} \text{ is where } H(\omega) = 0$$

(c) $\Rightarrow$ THIS MEANS THAT $s_1 = j\sqrt{3}$ and $s_2 = -j\sqrt{3}$

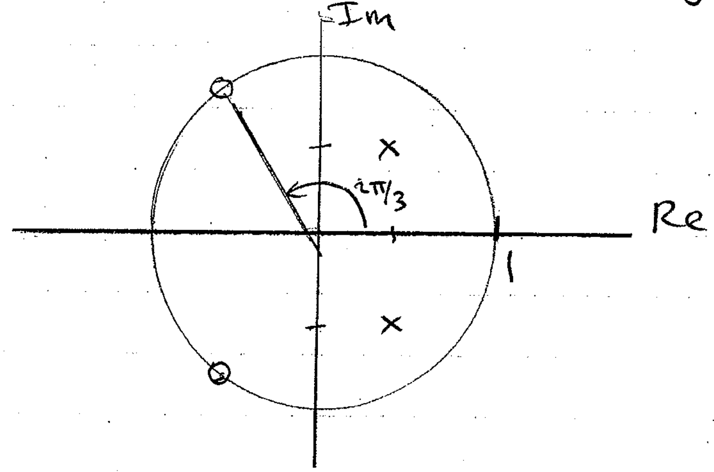
are mapped to $z_1 = e^{j\frac{2\pi}{3}}$ and $z_2 = e^{-j\frac{2\pi}{3}}$

(c) $-\frac{1}{5} + j\frac{2}{5}$ mapped to $p_1 = \dfrac{1 - \frac{1}{5} + j\frac{2}{5}}{1 + \frac{1}{5} - j\frac{2}{5}}$

$$= \frac{\frac{4}{5} + j\frac{2}{5}}{\frac{6}{5} - j\frac{2}{5}} = \frac{4+2j}{6-2j} = \frac{2+j}{3-j} \cdot \frac{3+j}{3+j}$$

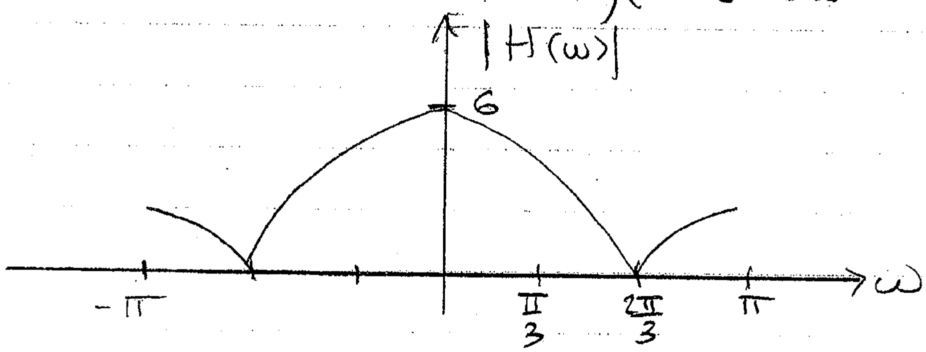
$$= \frac{(6-1) + j5}{10} = \frac{5+j5}{10} = \frac{1}{2} + \frac{1}{2}\hat{j}$$

(c) $p_1 = \frac{1}{2} + \frac{1}{2}j \qquad p_2 = \frac{1}{2} - \frac{1}{2}j$



(d)

$$H(z) = \frac{\left(z - e^{j\frac{2\pi}{3}}\right)\left(z - e^{-j\frac{2\pi}{3}}\right)}{\left(z - \frac{1}{2} + \frac{1}{2}j\right)\left(z - \frac{1}{2} - \frac{1}{2}j\right)}$$



(e) Note: $(z - p_1)(z - p_1^*) = z^2 - 2\,\mathrm{Re}(p_1)\,z + |p_1|^2$

$(z - z_1)(z - z_1^*) = z^2 - 2\,\mathrm{Re}(z_1)\,z + |z_1|^2$

$$y[n] - 2\left(\frac{1}{2}\right) y[n-1] + \left(\frac{1}{4} + \frac{1}{4}\right) y[n-2]$$

$$= x[n] - 2\cos\left(\frac{2\pi}{3}\right) x[n-1] + x[n-2]$$

zero on unit circle => magnitude = 1

$$y[n] = y[n-1] + \frac{1}{2} y[n-2] + x[n] + x[n-1] + x[n-2]$$