Continuous-Time Fourier Transform (Hz): \( X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt \)

Continuous-Time Fourier Transform Pair (Hz): \( \mathcal{F}\left\{ \frac{\sin(2\pi Wt)}{\pi t} \right\} = \text{rect} \left\{ \frac{F}{2W} \right\} \)

where \( \text{rect}(x) = 1 \) for \( |x| < 0.5 \) and \( \text{rect}(x) = 0 \) for \( |x| > 0.5 \).

Continuous-Time Fourier Transform Property: \( \mathcal{F}\{x_1(t)x_2(t)\} = X_1(F) * X_2(F) \),

where * denotes convolution, and \( \mathcal{F}\{x_i(t)\} = X_i(F), \ i = 1, 2 \).

Relationship between DTFT and CTFT frequency variables in Hz: \( \omega = 2\pi \frac{F}{F_s} \),

where \( F_s = \frac{1}{T_s} \) is the sampling rate in Hz.
(i) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y[n]$ of the system in Figure 1, when $x[n]$ is input to the system. Write output in sequence form (indicate where $n = 0$ is) OR do stem plot.

(ii) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_0[n] = x[n]*h_0[n]$, when $x[n]$ is input to the filter $h_0[n] = h[3n]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(iii) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_1[n] = x[n]*h_1[n]$, when $x[n]$ is input to the filter $h_1[n] = h[3n + 1]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(iv) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_2[n] = x[n]*h_2[n]$, when $x[n]$ is input to the filter $h_2[n] = h[3n + 2]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

Figure 3.

(d) Draw a block diagram of an efficient implementation of the filtering followed by downsampling system depicted in Fig. 3. Be sure to define all quantities.
Problem 1. Consider the upsampler system below in Figure 1.

Figure 1.

(a) Draw block diagram of an efficient implementation of the upsampler system in Fig. 1.

(b) Your answer to part (a) should involve the polyphase components of $h[n]$:
- $h_0[n] = h[3n]$, $h_1[n] = h[3n + 1]$, $h_2[n] = h[3n + 2]$. For the plots requested below, do all magnitude plots on one graph and you can do all phase plots on one graph.

- (i) For the ideal case where $h[n] = \frac{3 \sin(\frac{\pi n}{3})}{\pi n}$, plot the magnitude of the DTFT of $h_0[n] = h[3n]$, $H_0(\omega)$, over $-\pi < \omega < \pi$.

- (ii) For the general case where $h[n]$ is an arbitrary impulse response, express the DTFT of $h_1[n] = h[3n + 1]$, denoted $H_1(\omega)$, in terms of $H(\omega)$.

- (iii) For the ideal case where $h[n] = \frac{3 \sin(\frac{\pi n}{3})}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_1[n] = h[3n + 1]$, $H_1(\omega)$, over $-\pi < \omega < \pi$.

- (iv) For the general case where $h[n]$ is an arbitrary impulse response, express the DTFT of $h_2[n] = h[3n + 2]$, denoted $H_2(\omega)$, in terms of $H(\omega)$.

- (v) For the ideal case where $h[n] = \frac{3 \sin(\frac{\pi n}{3})}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_2[n] = h[3n + 2]$, $H_2(\omega)$, over $-\pi < \omega < \pi$.

(c) Consider that the input to the system in Figure 1 is a sampled version of the analog signal in Figure 2. For the remaining parts of this problem, the input signal is as defined below where $x_a(t)$ is the analog signal in Figure 2. Assume that $1/T_s = 1$ Hz is above the Nyquist rate for this signal. That is, even though this signal is not strictly bandlimited, assume that aliasing effects are negligible.

\[ x[n] = x_a(nT_s), \quad T_s = 1 \text{ sec} \]

Figure 2.
(b) 
\[ H_k(\omega) = \sum_{l=0}^{2} \left( e^{-\frac{\omega^2}{3}} \right) H \left( \frac{\omega - k2\pi}{3} \right) \] 
\[ \frac{\omega}{L} \]

(c) 
\[ y[n] = \begin{cases} 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, e^{-7}, e^{-5}, 4, 3, 2, 1, 0, \\ -1, -2, -3, -4, -5, -6, -7, -8, -9, -8, -7, -6, -5, -4, \\ -3, -2, -1, 0 \end{cases} \]

(c) 
\[ y[2n] = \{ 0, 3, 6, 9, 6, 3, 0, -2, -6, -9, -6, -3, 0 \} \]

(c) 
\[ y[2n+1] = \{ 1, 4, 7, 8, 5, 2, -1, -4, -7, -8, -5, -2, 0 \} \]

(c) 
\[ y[2n+3] = \{ 2, 5, 8, 7, 4, 1, -2, -5, -8, -7, -4, -1 \} \]
Problem 2. This problem is about digital subbanding of four different DT signals. For the sake of simplicity, the signals are the four infinite-length sinewave signals defined below.

\[ x_0[n] = \cos \left( \frac{\pi}{8} n \right) \quad x_1[n] = \cos \left( \frac{3\pi}{8} n \right) \quad x_2[n] = \cos \left( \frac{5\pi}{8} n \right) \quad x_3[n] = \cos \left( \frac{7\pi}{8} n \right) \]

Digital subbanding of these four signals is effected in an efficient way via the structure in Figure 4, where the various quantities are defined below: The impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response

\[ h_{LP}[n] = 4 \sin \left( \frac{\pi}{4} n \right) \pi n \]

\[ h^{+}_\ell[n] = h_{LP}[4n+\ell], \quad \ell = 0, 1, 2, 3. \]  

The respective signals at the inputs to these filters are formed from the input signals as described below, where \( \hat{x}_k[n] \) is the Hilbert Transform of \( x_k[n] \), \( k=0,1,2,3 \). (a) Plot the magnitude of the DTFT \( Y(\omega) \) of the interleaved signal \( y[n] \). Clearly indicate the frequencies of the sinewaves. (b) Draw a Block Diagram to recover the original signals, \( x_k[n], k = 0, 1, 2, 3 \), for the general case (not just for sinewaves.) You can denote the cosine matrix in Eq (3) as \( A \) and the sine matrix in Eq (3) as \( B \).

\[
\begin{bmatrix}
    y_0[n] \\
    y_1[n] \\
    y_2[n] \\
    y_3[n]
\end{bmatrix}
= \begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & \cos \left( \frac{2\pi}{4} (1) \right) & \cos \left( \frac{4\pi}{4} (1) \right) & \cos \left( \frac{6\pi}{4} (1) \right) \\
    1 & \cos \left( \frac{2\pi}{4} (2) \right) & \cos \left( \frac{4\pi}{4} (2) \right) & \cos \left( \frac{6\pi}{4} (2) \right) \\
    1 & \cos \left( \frac{2\pi}{4} (3) \right) & \cos \left( \frac{4\pi}{4} (3) \right) & \cos \left( \frac{6\pi}{4} (3) \right)
\end{bmatrix}
\begin{bmatrix}
    x_0[n] \\
    x_1[n] \\
    x_2[n] \\
    x_3[n]
\end{bmatrix}
+ \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & \sin \left( \frac{2\pi}{4} (1) \right) & \sin \left( \frac{4\pi}{4} (1) \right) & \sin \left( \frac{6\pi}{4} (1) \right) \\
    0 & \sin \left( \frac{2\pi}{4} (2) \right) & \sin \left( \frac{4\pi}{4} (2) \right) & \sin \left( \frac{6\pi}{4} (2) \right) \\
    0 & \sin \left( \frac{2\pi}{4} (3) \right) & \sin \left( \frac{4\pi}{4} (3) \right) & \sin \left( \frac{6\pi}{4} (3) \right)
\end{bmatrix}
\begin{bmatrix}
    \hat{x}_0[n] \\
    \hat{x}_1[n] \\
    \hat{x}_2[n] \\
    \hat{x}_3[n]
\end{bmatrix}
\]

\( A \)

\( B \)

\( (3) \)
Each sine wave is up sampled by a factor of \( \frac{3}{2} \).

(i) \( x_{0}^{\text{up}}[n] = \cos \left( \frac{\pi}{32} n \right) \)

(ii) \( x_{1}^{\text{up}}[n] = \cos \left( \frac{3\pi}{32} n \right) \Rightarrow \text{upper sideband starting at } \omega = \frac{\pi}{2} \)

\[ \Rightarrow \frac{n}{2} + \frac{3\pi}{32} = \frac{19\pi}{32} \]

(iii) \( x_{2}^{\text{up}}[n] = \cos \left( \frac{5\pi}{32} n \right) \Rightarrow \text{lower sideband working backwards from } \pi \)

\[ \Rightarrow \pi - \frac{5\pi}{32} = \frac{32\pi}{32} - \frac{5\pi}{32} = \frac{27\pi}{32} \]

(iv) \( x_{3}^{\text{up}}[n] = \cos \left( \frac{7\pi}{32} n \right) \Rightarrow \text{lower sideband working backwards from } \frac{\pi}{2} \)

\[ \Rightarrow \frac{\pi}{2} - \frac{7\pi}{32} = \frac{9\pi}{32} \]

\[ \Rightarrow \text{everything is real-valued, symmetric about } n = 0 \]

\[ \Rightarrow \text{negative frequencies are mirror image} \]

\( \text{with + sign in middle of Eqn (3), answer changes to Isb and freq= 13 } \pi /32 \)

\( \text{with + sign in middle of Eqn (3), answer changes to usb and freq= 23 } \pi /32 \)
(b) **Block Diagram**

\[
\begin{align*}
y(n) & \rightarrow h_0(n) \rightarrow z_0(n) \\
y(4n) & \rightarrow h_1(n) \rightarrow z_1(n) \\
y(4n+2) & \rightarrow h_2(n) \rightarrow z_2(n) \\
y(4n+3) & \rightarrow h_3(n) \rightarrow z_3(n)
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
z_0(n) \\
z_1(n) \\
z_2(n) \\
z_3(n)
\end{bmatrix} &= \begin{bmatrix}
x_0(n) \\
x_1(n) \\
x_2(n) \\
x_3(n)
\end{bmatrix} - B \begin{bmatrix}
\hat{x}_0(n) \\
\hat{x}_1(n) \\
\hat{x}_2(n) \\
\hat{x}_3(n)
\end{bmatrix}
\end{align*}
\]

\[
A^T B = 0
\]

\[
A^T A = \begin{bmatrix}
4 & 0 & 0 & 0 \\
0 & 2 & 0 & 2 \\
0 & 0 & 4 & 0 \\
0 & 2 & 0 & 2
\end{bmatrix}
\]

\[
B^T B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 \\
0 & 0 & 0 & 0 \\
0 & -2 & 0 & 2
\end{bmatrix}
\]

Thus:

\[
\begin{bmatrix}
u_0(n) \\
u_1(n) \\
u_2(n) \\
u_3(n)
\end{bmatrix} = A^T Z = \begin{bmatrix}
4x_0(n) \\
2x_1(n) + 2x_3(n) \\
4x_2(n) \\
2x_1(n) + 2x_3(n)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{y}_0(n) \\
\hat{y}_1(n) \\
\hat{y}_2(n) \\
\hat{y}_3(n)
\end{bmatrix} = B^T \hat{z} = \begin{bmatrix}
0 \\
2x_1(n) - 2x_3(n) \\
0 \\
-2x_1(n) + 2x_3(n)
\end{bmatrix}
\]

Run each of these third Hilbert Transformer to create \( \hat{y}_k(n) \).
\[^{\wedge} \gamma_1[n] = -2 \gamma_1[n] + 2 \gamma_3[n] \]
\[^{\wedge} \gamma_3[n] = 2 \gamma_1[n] - 2 \gamma_3[n] \]

Thus:
\[^{\wedge} \mu_0[n] = 4 \mu_0[n] \]
\[^{\wedge} \mu_1[n] - \gamma_1[n] = 4 \mu_1[n] \]
\[^{\wedge} \mu_2[n] = 4 \mu_2[n] \]
\[^{\wedge} \mu_3[n] - \gamma_3[n] = 4 \mu_3[n] \]

My vector \( z \) is the same thing as the original \( y \)
The most compact way to write the final answer is:

\[
\begin{bmatrix} T \\ T^\wedge \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} A \ y \ + \ B \ y \end{bmatrix}
\]

The plus sign + in the middle here would change to a minus sign - IF the plus sign + in the middle of Eqn 3 was changed to a minus sign -