SOLUTION

NAME: 26 Oct. 2018
ECE 538 Digital Signal Processing I Exam 2 Fall 2018

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET
Test Duration: 60 minutes.
Open Book but Closed Notes.
One (both sides) handwritten 8.5 in x 11 in crib sheet allowed
Calculators NOT allowed.
All work should be done in the space provided.

There are THREE problems.

Continuous-Time Fourier Transform (Hz): $X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

Continuous-Time Fourier Transform Pair (Hz): $\mathcal{F}\left\{\frac{\sin(2\pi Wt)}{\pi t}\right\} = rect\left\{\frac{f}{2W}\right\}$ where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = X_1(f) * X_2(f)$, where $*$ denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(f)$, $i = 1, 2$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x(t - t_0)\} = X(f)e^{-j2\pi ft_0}$, where $\mathcal{F}\{x(t)\} = X(f)$

Problem 1. Consider the upsampler system below in Figure 1.

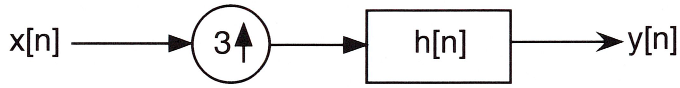


Figure 1.

(a) Draw a block diagram of an efficient implementation of the upsampler system in Fig. 1. Your answer to part (a) should involve the polyphase components of $h[n]$: $h_0[n] = h[3n]$, $h_1[n] = h[3n+1]$, and $h_2[n] = h[3n+2]$.

(b) Consider that the input to the system in Figure 1 is a sampled version of the analog Gaussian signal below sampled at a rate of $F_s = 4$ Hz. This is above Nyquist rate sampling, so no aliasing. **The answer to each of the parts below should be an expression that holds for all discrete-time.**

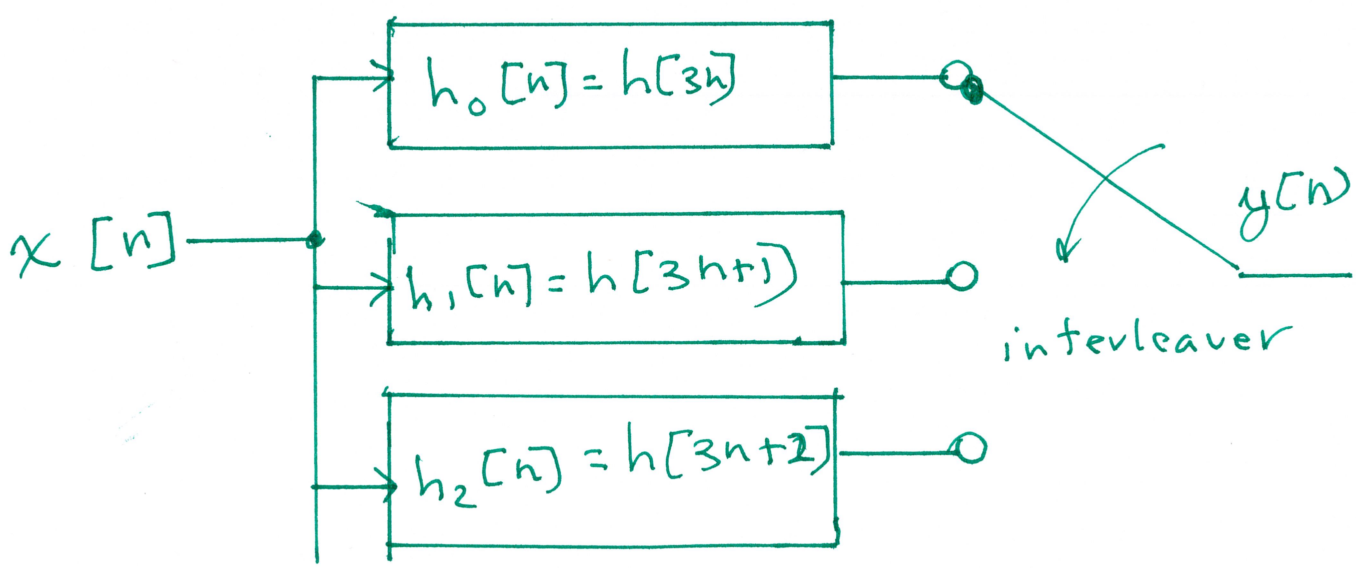
$$x[n] = x_a(nT_s), \qquad T_s = \frac{1}{4} \qquad \text{where:} \quad x_a(t) = e^{-\pi t^2}$$

(i) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_0[n] = x[n]*h_0[n]$, when $x[n]$ is input to the filter $h_0[n] = h[3n]$.

(ii) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_1[n] = x[n]*h_1[n]$, when $x[n]$ is input to the filter $h_1[n] = h[3n+1]$.

(iii) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_2[n] = x[n]*h_2[n]$, when $x[n]$ is input to the filter $h_2[n] = h[3n+2]$.

(iv) For the ideal case where $h[n] = 3\frac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y[n]$ of the system in Figure 1, when $x[n]$ is input to the system.



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(b)
(i) $h_0[n] = h[3n] = \dfrac{\sin(\pi n)}{\pi n} = \delta[n]$

$y_0[n] = x[n] = e^{-\pi t^2}\Big|_{t=\frac{n}{4}}$

$= e^{-\pi \frac{n^2}{16}}$

(ii) $h_1[n] = h[3n+1]$

$\Rightarrow y_1[n] = x_a\left(nT_s + \frac{T_s}{3}\right) = x_a\left(\frac{n}{4} + \frac{1}{12}\right)$

$= e^{-\pi\left(\frac{n}{4}+\frac{1}{12}\right)^2} = e^{-\pi \frac{(3n+1)^2}{144}}$

(iii) $h_2[n] = h[3n+2]$

$\Rightarrow y_2[n] = x_a\left(nT_s + \frac{2}{3}T_s\right) = x_a\left(\frac{n}{4} + \frac{2}{12}\right)$

$= e^{-\pi \frac{(3n+2)^2}{144}}$

(iv) $y[n] = x_a\left(n\frac{T_s}{3}\right) = x_a\left(\frac{n}{12}\right)$

$= e^{-\pi\left(\frac{n}{12}\right)^2} = e^{-\pi \frac{n^2}{144}}$

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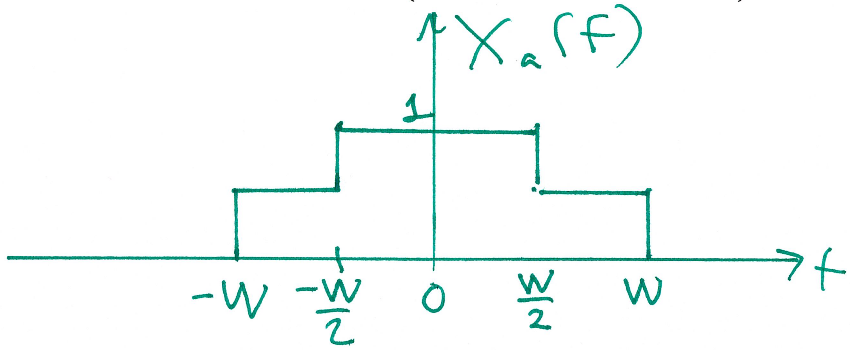
2(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

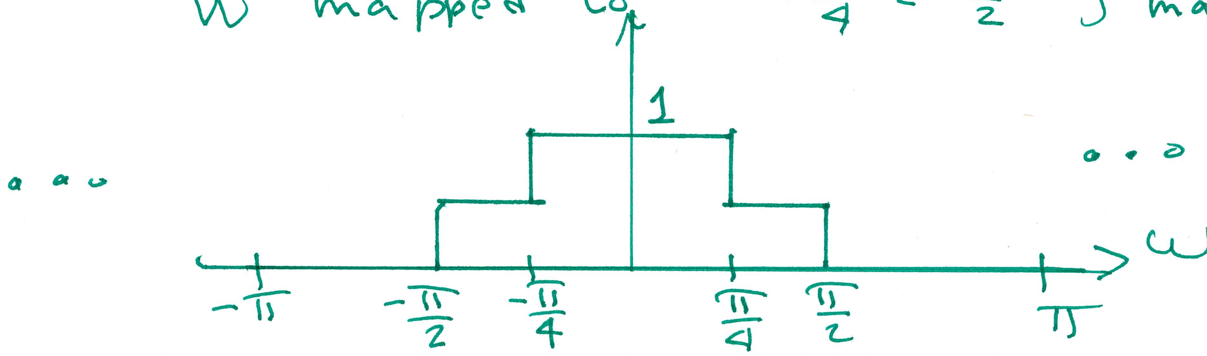
$$x_0(t) = T_s \frac{1}{2} \left\{ \frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{2} t)}{\pi t} \right\}$$



$F_s = 4W > 2W$ no aliasing

$\frac{W}{2}$ mapped to $\omega = \frac{2\pi\left(\frac{W}{2}\right)}{4W} = \frac{\pi}{4}$

W mapped to $2 \times \frac{\pi}{4} = \frac{\pi}{2}$ } it's a linear mapping

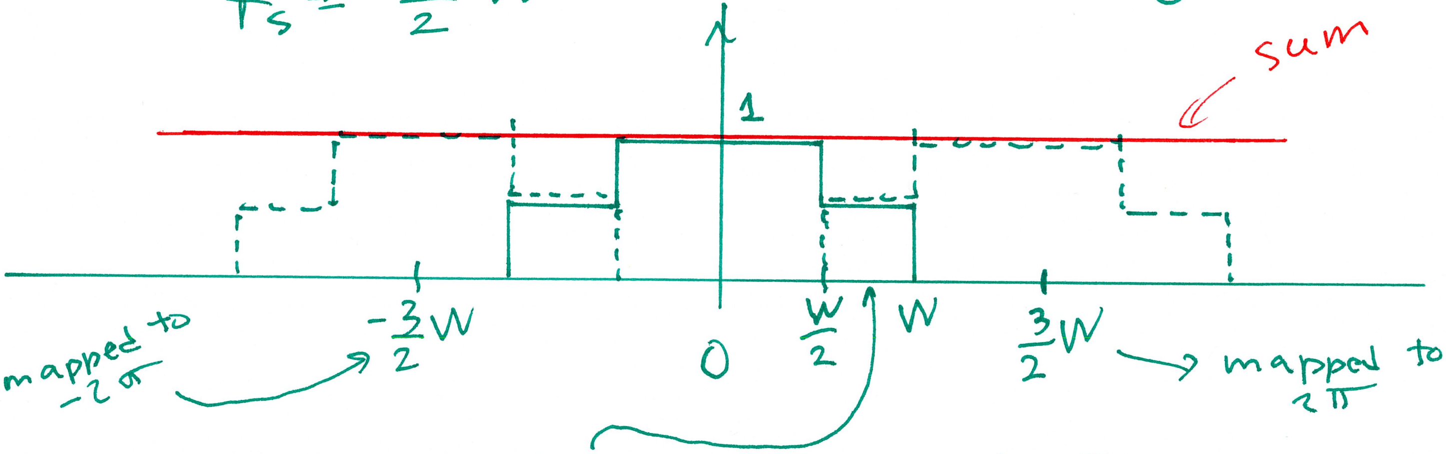


2(b) Consider the continuous-time signal $x_1(t)$ below. A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ with $F_s = \frac{1}{T_s} = \frac{3}{2}W$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_1(t) = T_s \frac{1}{2} \left\{ \frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{2} t)}{\pi t} \right\}$$

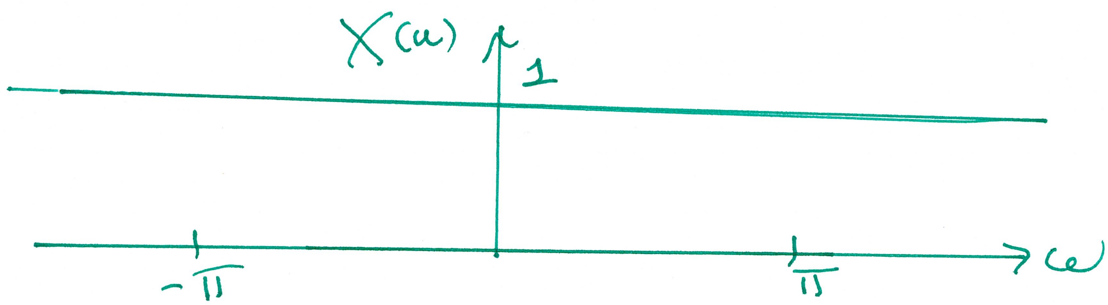
Same signal as in 2(a)

$F_s = \frac{3}{2} W < 2W \Rightarrow$ aliasing



$\frac{F_s}{2} = \frac{1}{2}\frac{3}{2} W = \frac{3}{4} W$ is mapped to π

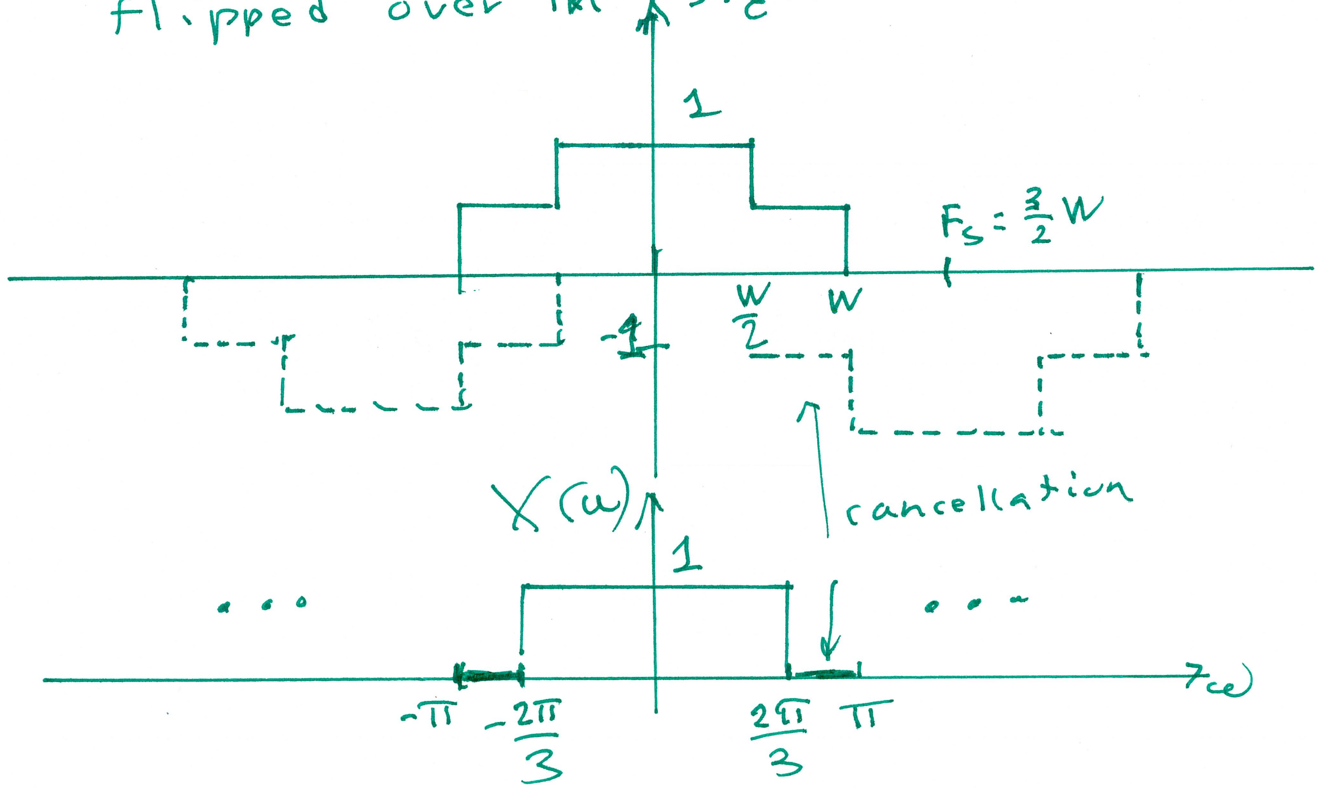
$\frac{W}{2}$ mapped to $\omega = 2\pi \frac{\frac{W}{2}}{\frac{3}{2}W} = \frac{2\pi}{3}$



2(c) Consider the continuous-time signal $x_2(t)$ below. A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2\left(nT_s + \frac{T_s}{2}\right)$ with $F_s = \frac{1}{T_s} = \frac{3}{2}W$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_2(t) = T_s \frac{1}{2}\left\{\frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{2} t)}{\pi t}\right\}$$

- same signal and sampling rate as 2(b) but "starting" at $\frac{T_s}{2}$ (or offset by $T_s/2$)
- as proved in class, the spectral replica at $+F_s$ and the one centered at $-F_s$ are flipped over in sign



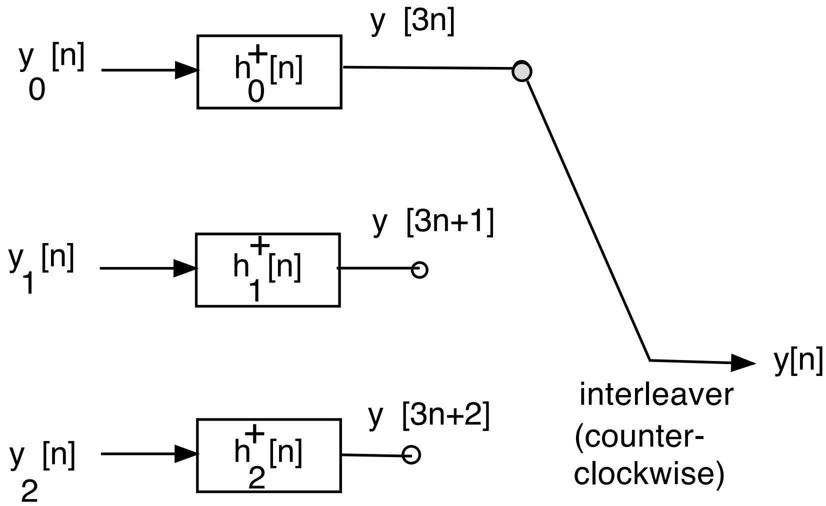


Figure 2.

Problem 3. This problem is about digital subbanding of the three DT signals $x_i[n]$, $i = 0, 1, 2$ defined below. Digital subbanding of these three signals is effected in an efficient way via filter bank in Figure 2. All of the quantities in Figure 2 are defined below; the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

$$h_{LP}[n] = 3\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 2 are defined as

$$h_\ell^+[n] = h_{LP}[3n+\ell], \quad \ell = 0, 1, 2. \tag{2}$$

The respective signals at the inputs to these filters are the signals below, all sampled at the Nyquist rate, $F_s = 2W$. That is $x_i[n] = x_i(nT_s)$, $i = 0, 1, 2$ where $T_s = \frac{1}{2W}$.

$$x_0(t) = T_s\frac{1}{2}\left\{\frac{\sin(2\pi W(t-t_0))}{\pi(t-t_0)} + \frac{\sin(2\pi W(t+t_0))}{\pi(t+t_0)}\right\} \quad \text{where:} \quad t_0 = \frac{1}{4W}$$

$$x_1(t) = T_s\frac{j}{2}\left\{\frac{\sin(2\pi W(t-t_0))}{\pi(t-t_0)} - \frac{\sin(2\pi W(t+t_0))}{\pi(t+t_0)}\right\} \quad \text{where:} \quad t_0 = \frac{1}{2W}$$

$$x_2(t) = T_s\frac{1}{2}\left\{\frac{1}{2}\frac{\sin(2\pi W(t-t_0))}{\pi(t-t_0)} + \frac{\sin(2\pi Wt)}{\pi t} + \frac{1}{2}\frac{\sin(2\pi W(t+t_0))}{\pi(t+t_0)}\right\} \quad \text{where:} \quad t_0 = \frac{1}{2W}$$

$$\begin{aligned}
y_0[n] = &\quad x_0[n] + \quad x_1[n]\cos\left(\tfrac{2\pi}{3}0\right) - \hat{x}_1[n]\sin\left(\tfrac{2\pi}{3}0\right) \quad + x_2[n]\cos\left(\tfrac{4\pi}{3}0\right) - \hat{x}_2[n]\sin\left(\tfrac{4\pi}{3}0\right)\\
y_1[n] = &\quad x_0[n] + \quad x_1[n]\cos\left(\tfrac{2\pi}{3}1\right) - \hat{x}_1[n]\sin\left(\tfrac{2\pi}{3}1\right) \quad + x_2[n]\cos\left(\tfrac{4\pi}{3}1\right) - \hat{x}_2[n]\sin\left(\tfrac{4\pi}{3}1\right)\\
y_2[n] = &\quad x_0[n] + \quad x_1[n]\cos\left(\tfrac{2\pi}{3}2\right) - \hat{x}_1[n]\sin\left(\tfrac{2\pi}{3}2\right) \quad + x_2[n]\cos\left(\tfrac{4\pi}{3}2\right) - \hat{x}_2[n]\sin\left(\tfrac{4\pi}{3}2\right)
\end{aligned} \tag{3}$$

Plot the magnitude of the DTFT $Y(\omega)$ of the interleaved signal $y[n]$ in Figure 2.

all three sampled at Nyquist Rate

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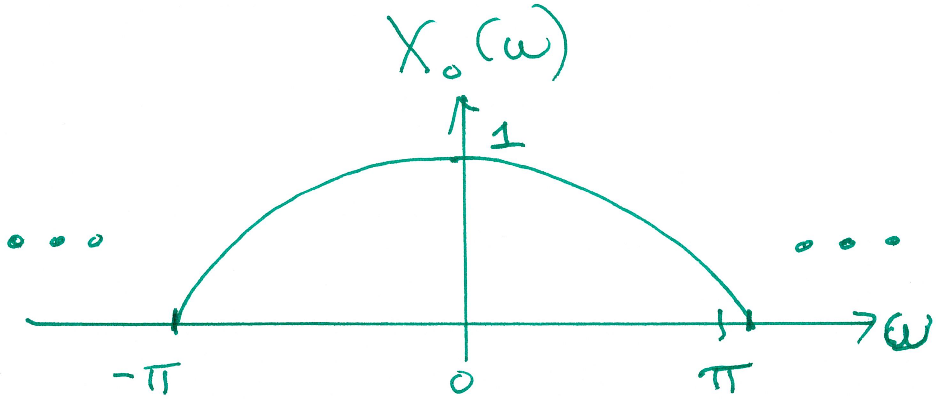
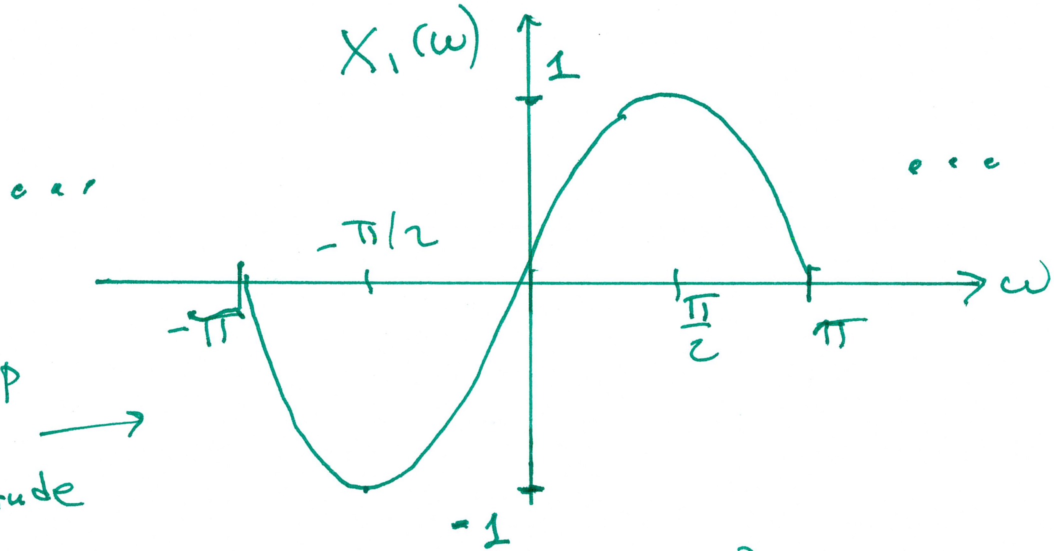
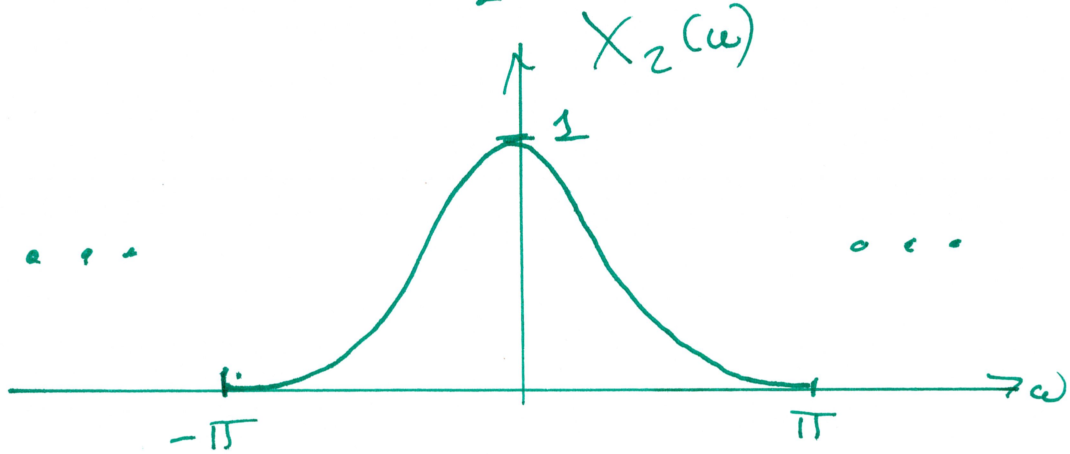




flip up to plot magnitude



full cycle of cosine but raised up by 1 to be positive => raised-cosine spectrum

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