SOLUTION

NAME: 27 Oct. 2017
ECE 538 Digital Signal Processing I Exam 2 Fall 2017

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET

Test Duration: 60 minutes.
Open Book but Closed Notes.
One (both sides) handwritten 8.5 in x 11 in crib sheet allowed
Calculators NOT allowed.
All work should be done in the space provided.

There are THREE problems.

Continuous-Time Fourier Transform (Hz): $X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$

Continuous-Time Fourier Transform Pair (Hz): $\mathcal{F}\left\{\frac{\sin(2\pi Wt)}{\pi t}\right\} = rect\left\{\frac{F}{2W}\right\}$ where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = X_1(F) * X_2(F)$, where * denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(F)$, $i = 1, 2$.

Relationship between DTFT and CTFT frequency variables in Hz: $\omega = 2\pi\frac{F}{F_s}$, where $F_s = \frac{1}{T_s}$ is the sampling rate in Hz

Problem 1. Consider the upsampler system below in Figure 1.

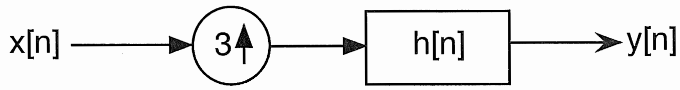

Figure 1.

(a) Draw block diagram of efficient implementation of the upsampler system in Fig. 1.

(b) Your answer to part (a) should involve the polyphase components of $h[n]$: $h_0[n] = h[3n]$, $h_1[n] = h[3n+1]$, and $h_2[n] = h[3n+2]$ and the DTFT of $h[n]$, denoted $H(\omega)$. **For the plots requested below, you can do all magnitude plots on one graph and you can do all phase plots on one graph, to save time and space.**

(i) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, plot the magnitude of the DTFT of $h_0[n] = h[3n]$, $H_0(\omega)$, over $-\pi < \omega < \pi$.

(ii) For the general case where $h[n]$ is an arbitrary impulse response, express the DTFT of $h_1[n] = h[3n+1]$, denoted $H_1(\omega)$, in terms of $H(\omega)$.

(iii) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_1[n] = h[3n+1]$, $H_1(\omega)$, over $-\pi < \omega < \pi$.

(iv) For the general case where $h[n]$ is an arbitrary impulse response, write an expression for the DTFT, $H_2(\omega)$, of $h_2[n] = h[3n+2]$ in terms of $H(\omega)$ that holds for all ω.

(v) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_2[n] = h[3n+2]$, $H_2(\omega)$, over $-\pi < \omega < \pi$.

(c) Consider that the input to the system in Figure 1 is a sampled version of the analog sinewave below (turned-on forever) sampled at a rate of $F_s = 2$ Hz. This is Nyquist rate sampling with no aliasing. **The answer to each of the parts below should be an expression that holds for all time, for example, a DT sinewave turned-on forever.**

$$x[n] = x_a(nT_s), \qquad T_s = \frac{1}{2} \qquad \text{where:} \quad x_a(t) = \cos(2\pi t)$$

(i) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_0[n] = x[n] * h_0[n]$, when $x[n]$ is input to the filter $h_0[n] = h[3n]$.

(ii) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_1[n] = x[n] * h_1[n]$, when $x[n]$ is input to the filter $h_1[n] = h[3n+1]$.

(iii) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y_2[n] = x[n] * h_2[n]$, when $x[n]$ is input to the filter $h_2[n] = h[3n+2]$.

(iv) For the ideal case where $h[n] = 3\dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$, determine the output $y[n]$ of the system in Figure 1, when $x[n]$ is input to the system.

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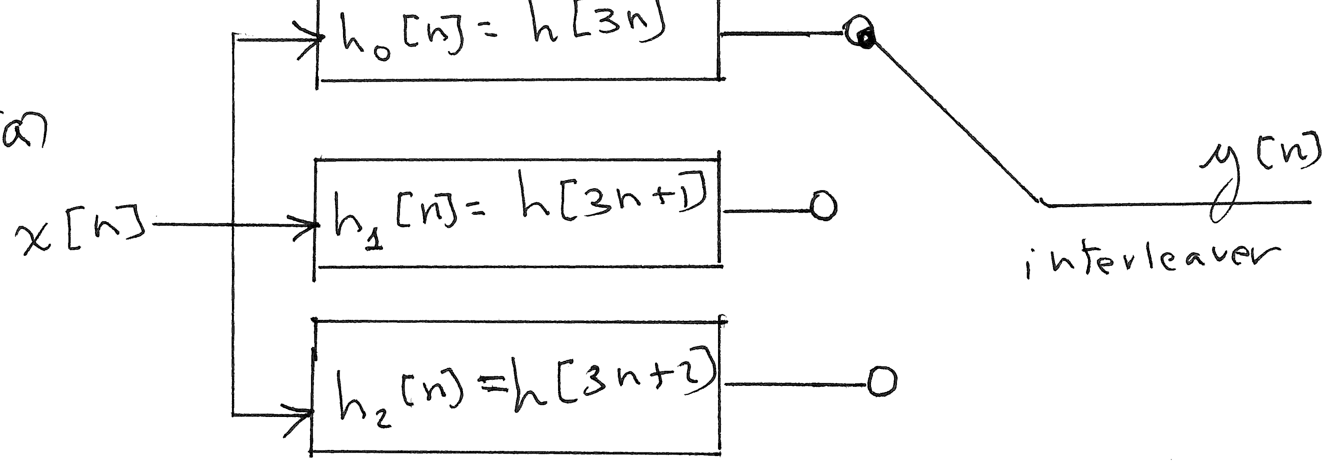
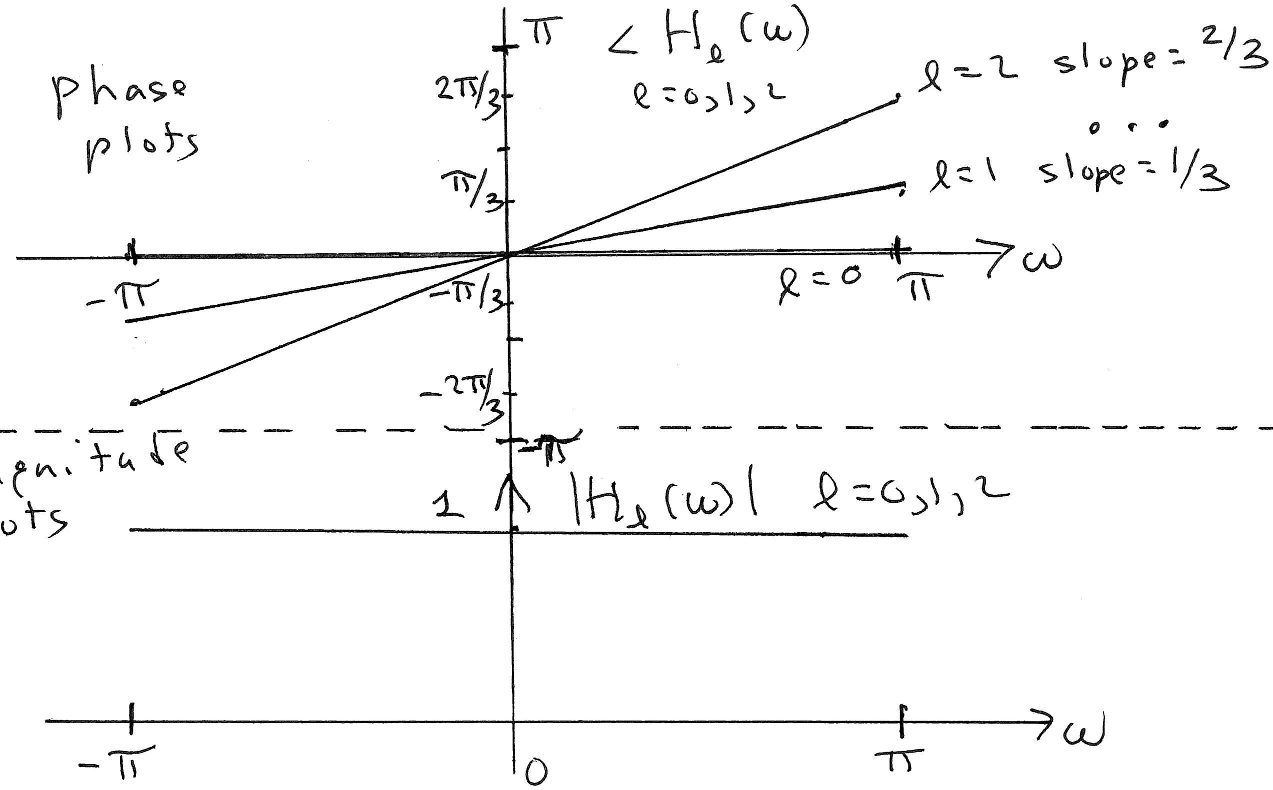
(a)



(b) ... phase plots

magnitude plots



(ii) $$H_1(\omega) = \frac{1}{3}\sum_{k=0}^{2} e^{j\frac{(\omega - k2\pi)}{3}} H\left(\frac{\omega - k2\pi}{3}\right)$$

(iv) $$H_2(\omega) = \frac{1}{3}\sum_{k=0}^{2} e^{j(\omega - k2\pi)\frac{2}{3}} H\left(\frac{\omega - k2\pi}{3}\right)$$

This page left intentionally blank for student work for Problem 1.

(c) - (i) $h_0[n] = \delta[n]$

$$y_0[n] = x[n] = \cos(2\pi t)\Big|_{t = n\frac{1}{2}} = \cos(\pi n)$$

(c) - (ii) $h_1[n] = \dfrac{\sin\left(\pi\left(n+\frac{1}{3}\right)\right)}{\pi\left(n+1/3\right)}$ $\quad -\infty < n < \infty$

$$y_1[n] = \cos\left(2\pi\left(n\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}\right)\right) = \cos\left(2\pi\left(\frac{n}{2}+\frac{1}{6}\right)\right)$$

$$= \cos\left(\pi\left(n+\frac{1}{3}\right)\right) = \cos\left(\pi n + \frac{\pi}{3}\right)$$

(c) - (iii) $h_2[n] = \dfrac{\sin\left(\pi\left(n+\frac{2}{3}\right)\right)}{\pi\left(n+\frac{2}{3}\right)}$ $\quad -\infty < n < \infty$

$$y_2[n] = \cos\left(2\pi\left(n\frac{1}{2} + \frac{2}{3}\left(\frac{1}{2}\right)\right)\right) = \cos\left(2\pi\left(\frac{n}{2}+\frac{1}{3}\right)\right)$$

$$= \cos\left(\pi\left(n+\frac{2}{3}\right)\right) = \cos\left(\pi n + \frac{2\pi}{3}\right)$$

(c) - (iv)

$$y[n] = x_a\left(n\frac{T_s}{3}\right) = x_a\left(n\frac{1}{3}\cdot\frac{1}{2}\right) = x_a\left(\frac{n}{6}\right)$$

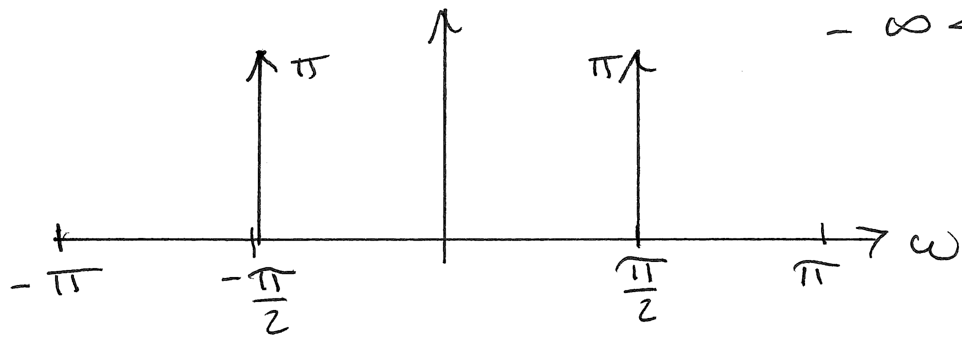
$$= \cos\left(2\pi\frac{n}{6}\right) = \cos\left(\frac{\pi}{3}n\right)$$

2(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_0(t) = \cos(2\pi W t)$$

$$x_0[n] = x_0(nT_s) = \cos\left(2\pi W \frac{n}{4W}\right) = \cos\left(\frac{\pi}{2} n\right)$$

$$-\infty < n < \infty$$



2(b) Consider the continuous-time signal $x_1(t)$ below. A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ with $F_s = \frac{1}{T_s} = 2W$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_1(t) = T_s \frac{1}{W}\left\{\frac{\sin(2\pi \frac{W}{2} t)}{\pi t}\right\}^2$$

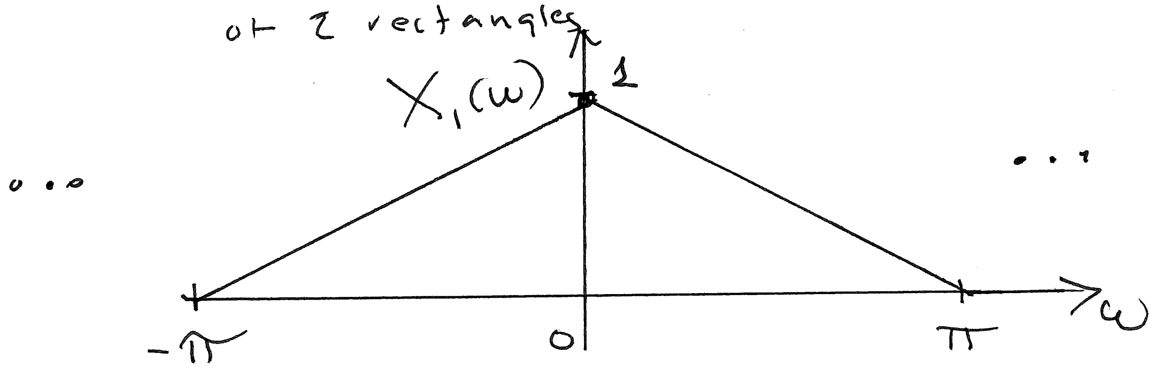
$$x_1[n] = \frac{1}{2W}\frac{1}{W}\left\{\frac{\sin\left(2\pi \frac{W}{2} \frac{n}{2W}\right)}{\pi \frac{n}{2W}}\right\}^2$$

$$= \frac{4}{2}\left\{\frac{\sin\left(\frac{\pi}{2} n\right)}{\pi n}\right\}^2$$

height of triangle in frequency domain:

$$2 \quad \frac{1}{2\pi}\left\{1^2 \left(2\frac{\pi}{2}\right)\right\} = \frac{1}{\pi}\pi = 1$$

height width
of product for
$\omega = 0$ in convolution
of 2 rectangles

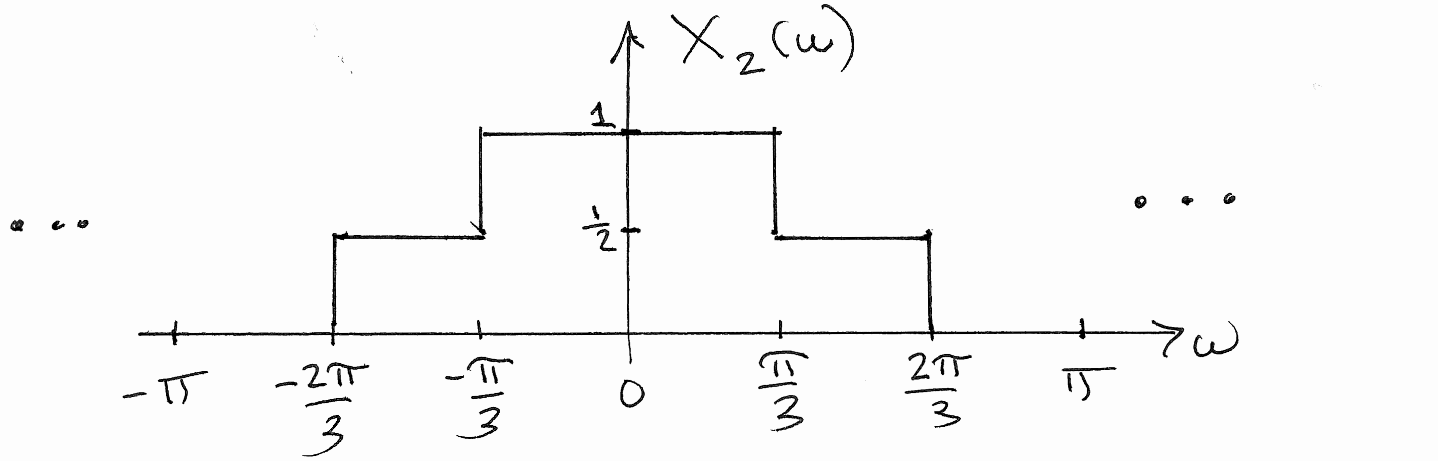


2(c) Consider the continuous-time signal $x_2(t)$ below. A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2(nT_s)$ with $F_s = \frac{1}{T_s} = 3W$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_2(t) = T_s \frac{1}{2} \left\{ \frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{2} t)}{\pi t} \right\}$$

$$x_2[n] = \frac{1}{3W} \frac{1}{2} \left\{ \frac{\sin\left(2\pi W \frac{n}{3W}\right)}{\pi \frac{n}{3W}} + \frac{\sin\left(2\pi \frac{W}{2} \frac{n}{3W}\right)}{\pi \frac{n}{3W}} \right\}$$

$$= \frac{1}{2} \left\{ \frac{\sin\left(\frac{2\pi}{3} n\right)}{\pi n} + \frac{\sin\left(\frac{\pi}{3} n\right)}{\pi n} \right\} \quad -\infty < n < \infty$$



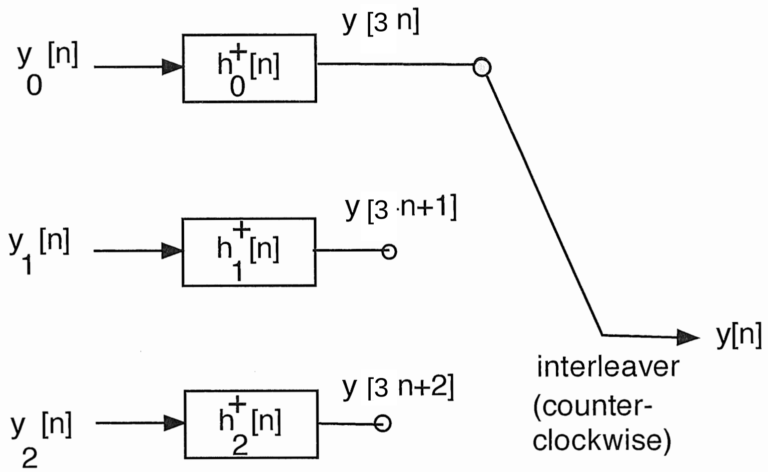


Figure 2.

Problem 3. This problem is about digital subbanding of the three DT signals $x_i[n]$, $i = 0, 1, 2$ from Problem 2. Digital subbanding of these three signals is effected in an efficient way via filter bank in Figure 2. All of the quantities in Figure 2 are defined below; the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

$$h_{LP}[n] = 3\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 2 are defined as

$$h_\ell^+[n] = h_{LP}[3n + \ell], \quad \ell = 0, 1, 2. \tag{2}$$

The respective signals at the inputs to these filters are formed from the input signals as (from Problem 2) as described below. **There is only ONE part to this problem: plot the magnitude of the DTFT $Y(\omega)$ of the interleaved signal $y[n]$.**

$$\begin{aligned}
y_0[n] &= x_0[n] + x_1[n]\cos\left(\tfrac{2\pi}{3}0\right) - \hat{x}_1[n]\sin\left(\tfrac{2\pi}{3}0\right) + x_2[n]\cos\left(\tfrac{2\pi}{3}0\right) + \hat{x}_2[n]\sin\left(\tfrac{2\pi}{3}0\right)\\
y_1[n] &= x_0[n] + x_1[n]\cos\left(\tfrac{2\pi}{3}1\right) - \hat{x}_1[n]\sin\left(\tfrac{2\pi}{3}1\right) + x_2[n]\cos\left(\tfrac{2\pi}{3}1\right) + \hat{x}_2[n]\sin\left(\tfrac{2\pi}{3}1\right)\\
y_2[n] &= x_0[n] + x_1[n]\cos\left(\tfrac{2\pi}{3}2\right) - \hat{x}_1[n]\sin\left(\tfrac{2\pi}{3}2\right) + x_2[n]\cos\left(\tfrac{2\pi}{3}2\right) + \hat{x}_2[n]\sin\left(\tfrac{2\pi}{3}2\right)
\end{aligned} \tag{3}$$

NAME:

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