SOLUTION

NAME: **28 Oct. 2016**

ECE 538 Digital Signal Processing I Exam 2 Fall 2016

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET

Test Duration: 60 minutes.

Open Book but Closed Notes.

One (both sides) handwritten 8.5 in x 11 in crib sheet allowed

Calculators NOT allowed.

All work should be done in the space provided.

There are three problems.

1. 50 points
2. 30 points
3. 20 points

Continuous-Time Fourier Transform (Hz): $X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$

Continuous-Time Fourier Transform Pair (Hz): $\mathcal{F}\left\{\frac{\sin(2\pi Wt)}{\pi t}\right\} = rect\left\{\frac{F}{2W}\right\}$ where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = X_1(F) * X_2(F)$, where $*$ denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(F)$, $i = 1, 2$.

Relationship between DTFT and CTFT frequency variables in Hz: $\omega = 2\pi\frac{F}{F_s}$, where $F_s = \frac{1}{T_s}$ is the sampling rate in Hz

Problem 1. Consider the upsampler system below in Figure 1.

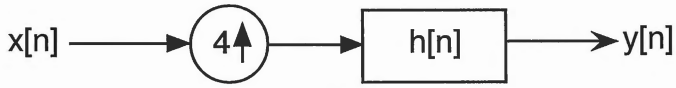


Figure 1.

(a) Draw block diagram of efficient implementation of the upsampler system in Fig. 1.

(b) Your answer to part (a) should involve the polyphase components of $h[n]$: $h_0[n] = h[4n]$, $h_1[n] = h[4n+1]$, $h_2[n] = h[4n+2]$, and $h_3[n] = h[4n+3]$. For the plots requested below, you can do all magnitude plots on one graph and you can do all phase plots on one graph, to save time and space.

(i) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, plot the magnitude of the DTFT of $h_0[n] = h[4n]$, $H_0(\omega)$, over $-\pi < \omega < \pi$.

(ii) For the general case where $h[n]$ is an arbitrary impulse response, express the DTFT of $h_1[n] = h[4n+1]$, denoted $H_1(\omega)$, in terms of $H(\omega)$.

(iii) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_1[n] = h[4n+1]$, $H_1(\omega)$, over $-\pi < \omega < \pi$.

(iv) Express the DTFT of $h_2[n] = h[4n+2]$, denoted $H_2(\omega)$, in terms of $H(\omega)$.

(v) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_2[n] = h[4n+2]$, $H_2(\omega)$, over $-\pi < \omega < \pi$.

(vi) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, plot both the magnitude AND phase of the DTFT $h_3[n] = h[4n+3]$, $H_3(\omega)$, over $-\pi < \omega < \pi$.

(c) Consider that the input to the system in Figure 1 is a sampled version of the analog signal in Figure 2. For the remaining parts of this problem, the input signal is as defined below where $x_a(t)$ is the analog signal in Figure 2. Assume that $1/T_s = 1/2$ is above the Nyquist rate for this signal. That is, even though this signal is not strictly bandlimited, assume that aliasing effects are negligible.

$$x[n] = x_a(nT_s), \qquad T_s = 2$$

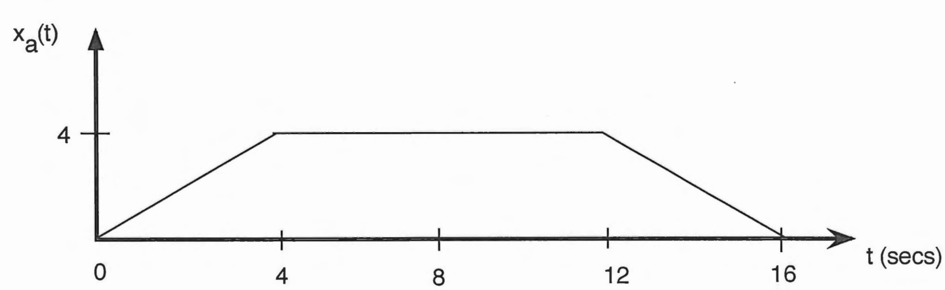


Figure 2.

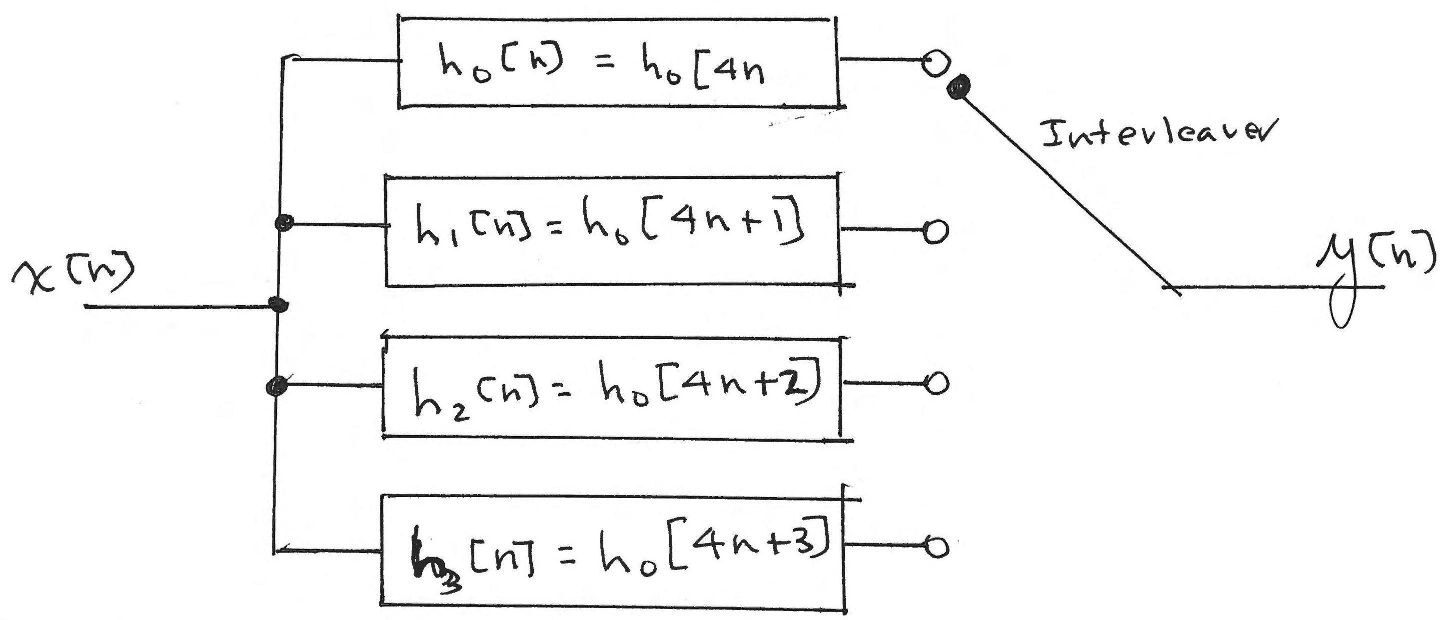
(i) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, determine the output $y[n]$ of the system in Figure 1, when $x[n]$ is input to the system. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(ii) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, determine the output $y_0[n] = x[n]*h_0[n]$, when $x[n]$ is input to the filter $h_0[n] = h[4n]$. Write output in sequence form (indicating where is $n = 0$ OR do stem plot.

(iii) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, determine the output $y_1[n] = x[n]*h_1[n]$, when $x[n]$ is input to the filter $h_1[n] = h[4n+1]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(iv) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, determine the output $y_2[n] = x[n]*h_2[n]$, when $x[n]$ is input to the filter $h_2[n] = h[4n+2]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.

(v) For the ideal case where $h[n] = 4\frac{\sin(\frac{\pi}{4}n)}{\pi n}$, determine the output $y_3[n] = x[n]*h_3[n]$, when $x[n]$ is input to the filter $h_3[n] = h[4n+3]$. Write output in sequence form (indicating where $n = 0$ is) OR do stem plot.



For $\ell = 0, 1, 2, 3$:

$$H_\ell(\omega) = \left\{ \frac{1}{4} \sum_{k=0}^{3} e^{-jk\frac{2\pi\ell}{4}} H\left(\frac{\omega - k2\pi}{4}\right) \right\} e^{j\frac{\ell}{4}\omega}$$

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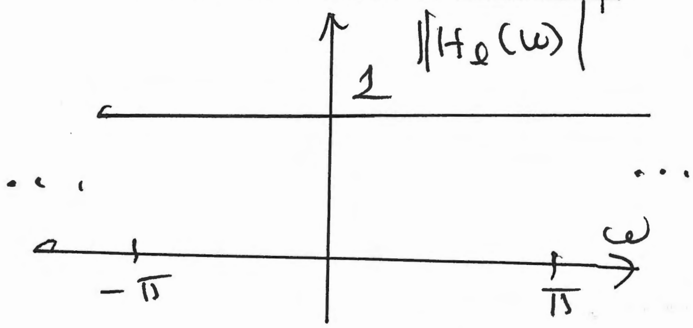
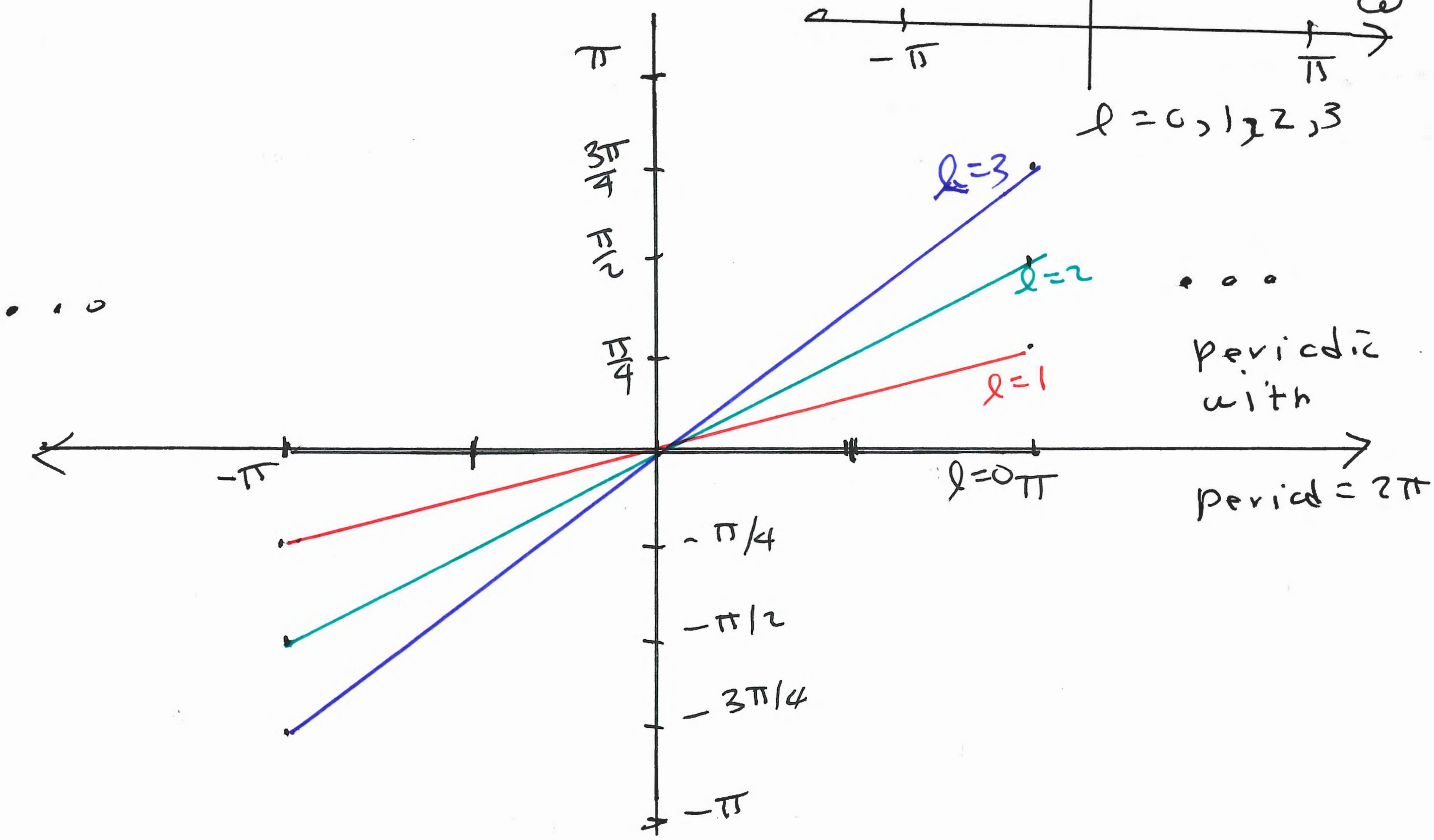
For the ideal case where: $\boxed{h[n] = \dfrac{\sin\left(\frac{\pi}{4}n\right)}{\frac{\pi}{4}n}}$



$\ell = 0, 1, 2, 3$

· Then: for $-\pi < \omega < \pi$:

$$H_\ell(\omega) = e^{j\frac{\ell}{4}\omega}$$



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1(c) $x[n] = x_a(n2) \Rightarrow$ sample every 2 secs.

$$x[n] = \{\underset{\uparrow}{0}, 2, 4, 4, 4, 4, 4, 2, 0\}$$

(i) digital upsampling by a factor of 4: $\Rightarrow$ effectively $\Rightarrow$ sample every $\frac{2}{4} = \frac{1}{2}$ sec.

blue blue

$$y[n] = \{\underset{\uparrow}{0}, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,$$
$$4, 4, 4, 4, 4, \tfrac{7}{2}, 3, \tfrac{5}{2}, 2, \tfrac{3}{2}, 1, \tfrac{1}{2}, 0\}$$

(ii) $h_0[n] = h[4n] = \dfrac{\sin(\pi n)}{\pi n} = \delta[n] \Rightarrow$ samples you already have

$y_0[n] = x[n] = \{\underset{\uparrow}{0}, 2, 4, 4, 4, 4, 4, 2, 0\}$ black

(iii) $h_1[n] = h[4n+1] = x_a\left(\frac{T_s}{4} + nT_s\right) = x_a\left(\frac{1}{2} + n2\right)$

$= y[4n+1] =$

$= \text{red} = \{\underset{\uparrow}{\tfrac{1}{2}}, \tfrac{5}{2}, 4, 4, 4, 4, \tfrac{7}{2}, \tfrac{3}{2}\}$

(iv) $h_2[n] = h[4n+2]$ $= x_a\left(\frac{2T_s}{4} + nT_s\right)$

$\Rightarrow y[4n+2] = \text{green} = \{\underset{\uparrow}{1}, 3, 4, 4, 4, 4, 3, 1\} = x_a(1+n2)$

(v) $h_3[n] = h[4n+3]$

$\Rightarrow y[4n+3] = \text{blue} = \{\underset{\uparrow}{\tfrac{3}{2}}, \tfrac{7}{2}, 4, 4, 4, 4, \tfrac{5}{2}, \tfrac{1}{2}\}$

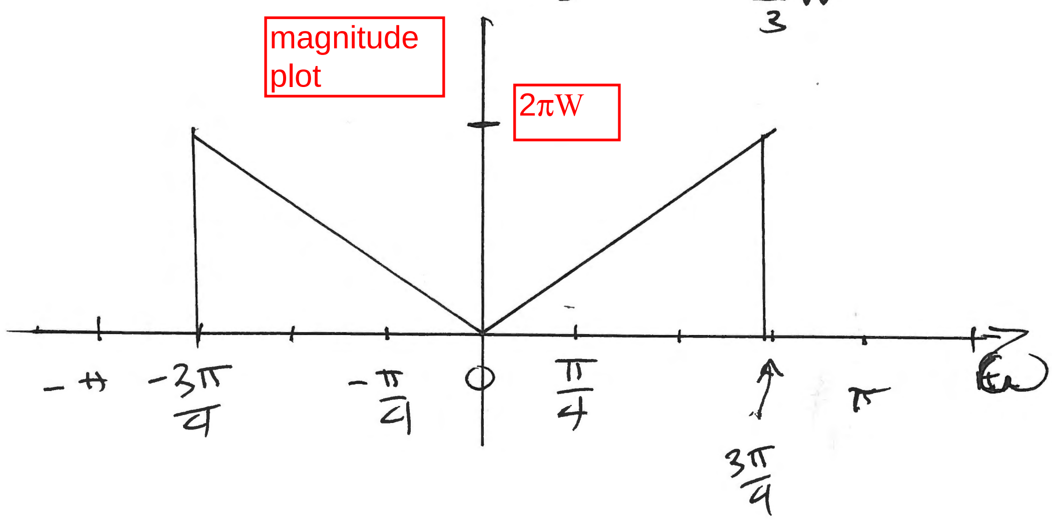
$= x_a\left(\frac{3T_s}{4} + nT_s\right) = x_a\left(\frac{3}{2} + n2\right)$

Prob. 2(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ with $F_s = \frac{1}{T_s} = \frac{8}{3}W$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_0(t) = T_s \frac{d}{dt}\left\{\frac{\sin(2\pi W t)}{\pi t}\right\}$$

$$X_a(F) = T_s \, j2\pi F \, \mathrm{rect}\left(\frac{F}{2W}\right)$$

W mapped to $2\pi \frac{W}{F_s} = 2\pi \frac{W}{\frac{8}{3}W} = \frac{3}{4}\pi$

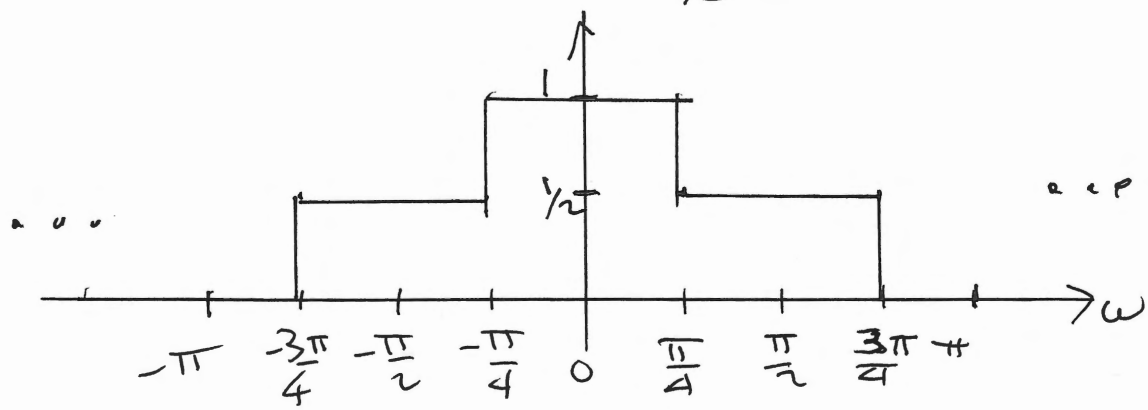


2(b) Consider the continuous-time signal $x_1(t)$ below. A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ with $F_s = \frac{1}{T_s} = \frac{8}{3}W$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_1(t) = T_s \frac{1}{2}\left\{\frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{3} t)}{\pi t}\right\}$$

W mapped to $\frac{3}{4}\pi$ from 2(a)

$\frac{W}{3}$ mapped to $2\pi \frac{W}{\frac{8}{3}W} = \frac{1}{3} \cdot \frac{3}{4}\pi = \frac{\pi}{4}$



2(c) Consider the continuous-time signal $x_2(t)$ below. A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_2(t) = T_s \left\{ \frac{\sin(2\pi \frac{W}{4} t)}{\pi t} \frac{\sin(2\pi \frac{3W}{4} t)}{\pi t} \right\} \cos(2\pi W t)$$

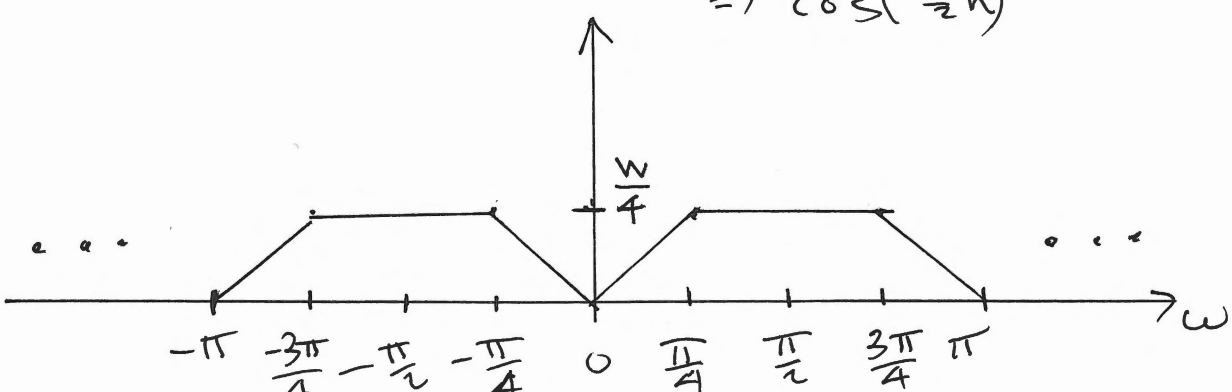
$\frac{3W}{4} - \frac{W}{4} = \frac{W}{2}$ $\qquad$ $\frac{W}{4} + \frac{3W}{4} = W$

mapped to: (before multiplying by cosine)

$\frac{2\pi \frac{W}{2}}{4W} = \frac{\pi}{4}$ $\qquad$ $2\pi \frac{W}{4W} = \frac{\pi}{2}$

$\cos(2\pi Wt)$ mapped to $2\pi \frac{W}{4W} = \frac{\pi}{2}$

$\Rightarrow \cos(\frac{\pi}{2} n)$

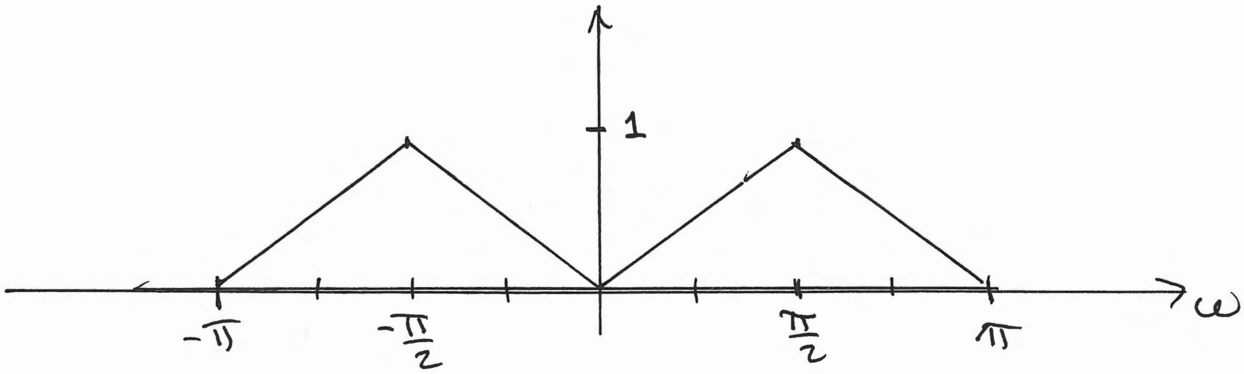


$$x[n] = \frac{1}{4W} \frac{\sin(\frac{\pi}{8} n)}{\pi \frac{n}{4W}} \cdot \frac{\sin(\frac{3\pi}{8} n)}{\pi \frac{n}{4W}} \cos(\frac{\pi}{2} n)$$

$$\text{height} = 4W \frac{1}{2} \cdot \frac{\pi}{4} \cdot \frac{1}{2\pi} = \frac{W}{4}$$

2(d) Consider the continuous-time signal $x_3(t)$ below. A discrete-time signal is created by sampling $x_3(t)$ according to $x_3[n] = x_3(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_3[n]$, $|X_3(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_3(t) = T_s \frac{2}{W} \left\{ \frac{\sin\left(2\pi \frac{W}{2} t\right)}{\pi t} \right\}^2 \cos(2\pi W t)$$



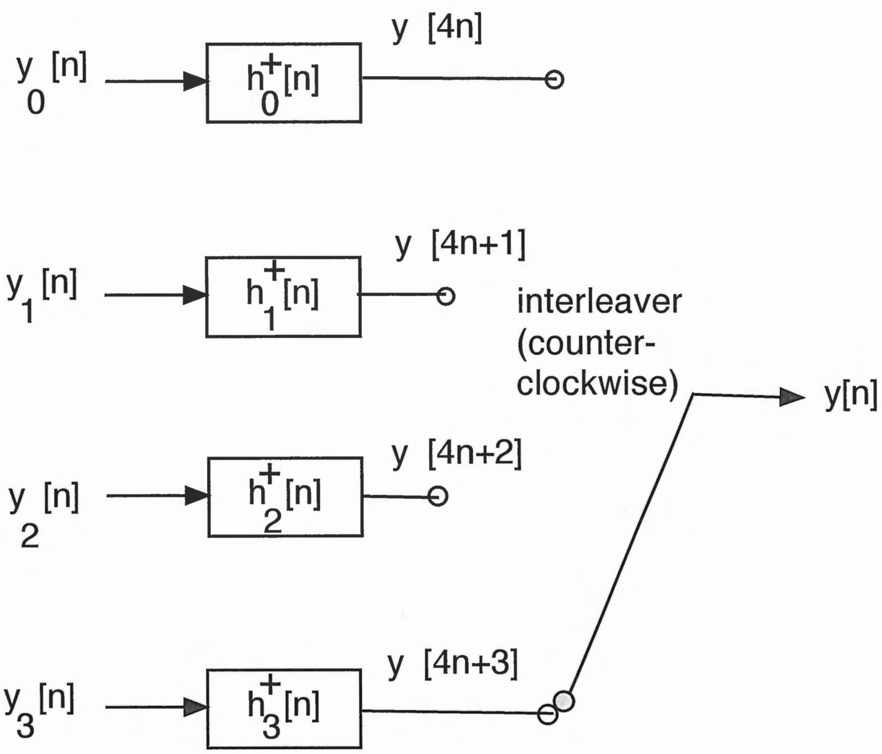


Figure 3.

Problem 2. This problem is about digital subbanding of the four DT signals $x_i[n]$, $i = 0, 1, 2, 3$ from Problem 2. Digital subbanding of these four signals is effected in an efficient way via filter bank in Figure 3. All of the quantities in Figure 1 are defined below: the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

$$h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 3 are defined as

$$h_\ell^+[n] = h_{LP}[4n+\ell], \qquad \ell = 0, 1, 2, 3. \tag{2}$$

The respective signals at the inputs to these filters are formed from the input signals as (from Problem 3) as described below. **There is only ONE part to this problem: plot the magnitude of the DTFT $Y(\omega)$ of the interleaved signal $y[n]$.**

$$\begin{aligned}
y_0[n] = &\ x_0[n] + x_1[n]\cos\left(\tfrac{\pi}{2}0\right) - \hat{x}_1[n]\sin\left(\tfrac{\pi}{2}0\right) + x_2[n]\cos\left(\tfrac{\pi}{2}0\right) + \hat{x}_2[n]\sin\left(\tfrac{\pi}{2}0\right) + x_3[n]\cos(\pi 0)\\
y_1[n] = &\ x_0[n] + x_1[n]\cos\left(\tfrac{\pi}{2}1\right) - \hat{x}_1[n]\sin\left(\tfrac{\pi}{2}1\right) + x_2[n]\cos\left(\tfrac{\pi}{2}1\right) + \hat{x}_2[n]\sin\left(\tfrac{\pi}{2}1\right) + x_3[n]\cos(\pi 1)\\
y_2[n] = &\ x_0[n] + x_1[n]\cos\left(\tfrac{\pi}{2}2\right) - \hat{x}_1[n]\sin\left(\tfrac{\pi}{2}2\right) + x_2[n]\cos\left(\tfrac{\pi}{2}2\right) + \hat{x}_2[n]\sin\left(\tfrac{\pi}{2}2\right) + x_3[n]\cos(\pi 2)\\
y_3[n] = &\ x_0[n] + x_1[n]\cos\left(\tfrac{\pi}{2}3\right) - \hat{x}_1[n]\sin\left(\tfrac{\pi}{2}3\right) + x_2[n]\cos\left(\tfrac{\pi}{2}3\right) + \hat{x}_2[n]\sin\left(\tfrac{\pi}{2}3\right) + x_3[n]\cos(\pi 3)
\end{aligned} \tag{3}$$

DTFT of each signal is compressed by a factor of 4

Signal ϕ is centered at $\omega = 0$

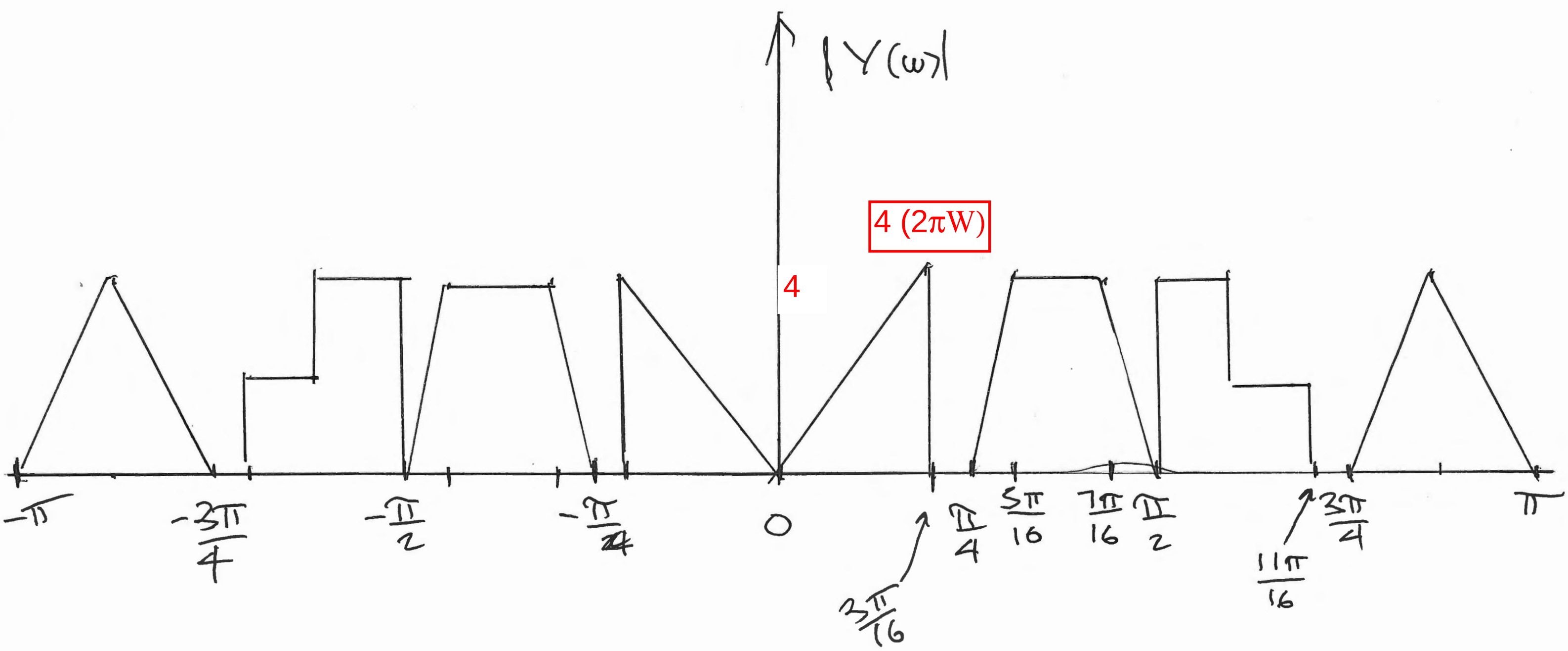
Signal 3 is centered at $\omega = \pi$ (and $\omega = -\pi$)

Signal 1's upper sideband is in $\frac{\pi}{2} < \omega < \frac{3\pi}{4}$

mirror image in $-\frac{3\pi}{4} < \omega < -\frac{\pi}{2}$

Signal 2's lower sideband is in $\frac{\pi}{4} < \omega < \frac{\pi}{2}$

mirror image in $-\frac{\pi}{2} < \omega < -\frac{\pi}{4}$



NAME:

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