NAME: 30 Oct. 2015
ECE 538 Digital Signal Processing I Exam 2 Fall 2015

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET
Test Duration: 60 minutes.
Open Book but Closed Notes.
One (both sides) handwritten 8.5 in x 11 in crib sheet allowed
Calculators NOT allowed.
All work should be done in the space provided.

There are two problems.
Problem 1 has 4 parts, 1(a) thru 1(d).
Problem 2 has 6 parts, 2(a) thru 2(f).

Continuous-Time Fourier Transform (Hz): \( X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt \)

Continuous-Time Fourier Transform Pair (Hz): \( \mathcal{F}\left\{ \frac{\sin(2\pi Wt)}{\pi t} \right\} = \text{rect}\left\{ \frac{F}{2W} \right\} \) where \( \text{rect}(x) = 1 \) for \( |x| < 0.5 \) and \( \text{rect}(x) = 0 \) for \( |x| > 0.5 \).

Continuous-Time Fourier Transform Property: \( \mathcal{F}\{x_1(t)x_2(t)\} = X_1(F) * X_2(F) \), where \( * \) denotes convolution, and \( \mathcal{F}\{x_i(t)\} = X_i(F) \), \( i = 1, 2 \).

Relationship between DTFT and CTFT frequency variables in Hz: \( \omega = 2\pi \frac{F}{F_s} \), where \( F_s = \frac{1}{T_s} \) is the sampling rate in Hz.
Prob. 1(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ with $F_s = \frac{1}{T_s} = 8W$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work. **NOTE:** The signal $x_0[n]$ is the input signal for each of the three remaining parts of this problem.

$$x_0(t) = T_s \frac{1}{2W} \frac{\sin(2\pi W t)}{\pi t} \frac{\sin(2\pi 3W t)}{\pi t}$$

- Maximum frequency $W + 3W$ is mapped to:
  $$\omega = 2\pi \frac{4W}{8W} = \pi$$

- End of flat part $3W - W = 2W$ mapped to:
  $$\omega = 2\pi \frac{2W}{4W} = \frac{\pi}{2}$$

- $X_0(\omega)$

- $F_s^{(o)} = 8W$  $T_s^{(o)} = \frac{1}{8W}$
1(b) The discrete-time signal, \( x_1[n] \), is created by running \( x_0[n] \) from part (a) thru the DT system below. You don’t have to do a lot of work but clearly explain your answers.

\[
h_1[n] = 3 \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}
\]

(i) Plot the magnitude of the DTFT of \( x_1[n] \), \( |X_1(\omega)| \), over \(-\pi < \omega < \pi\).

(ii) What is the new effective sampling rate, \( F_s^{(1)} \), at the output relative to original sampling rate \( F_s = 8W \)?

- no aliasing, so

\[
F_s^{(1)}_{\text{new}} = \frac{3}{2} \quad F_s = \frac{3}{2} \cdot 8W = 12W
\]

\[
T_s^{(1)} = \frac{1}{12W}
\]

- new max freq. in DT frequency domain:

\[
\omega_M = \frac{2}{3} \pi = 2\pi \frac{4W}{12W} = \frac{2}{3} \pi
\]

- The decimation by 2 causes a gain reduction by 2 (an amplitude)

combined with filter gain of 3 \( \Rightarrow \) new height = \( \frac{3}{2} = 1.5 \)

\[ X_1(\omega) \uparrow 1.5 \]

\[ \cdots \quad -\frac{2}{3}\pi \quad 0 \quad \frac{2}{3}\pi \quad \pi \quad \cdots \]

\[ \omega \]
1(c) The discrete-time signal, $x_2[n]$, is created by running $x_0[n]$ from part (a) through the DT system below.

$$h_2[n] = 2 \frac{\sin \left( \frac{\pi}{2} n \right)}{\pi n}$$

(i) Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$.

(ii) Express the new effective sampling rate, $F_s^{(2)}$, at the output in terms of the original sampling rate $F_s = 8W$.

- This is upsampling by a factor of 2

$$F_s^{(2)} = 2 F_s = 2 (8W) = 16 W \quad T_s^{(2)} = \frac{1}{16W}$$

$$X_2(\omega) = X_0(2\omega) H_2(\omega)$$

$= \text{plot above}$
1(d) The discrete-time signal, $x_3[n]$, is created by running $x_0[n]$ through the DT system below; "the output" refers to $x_3[n]$ in all parts below.

\[ h_3[n] = 3 \frac{\sin\left(\frac{\pi}{3} n\right)}{\pi n} \]

(i) Plot the magnitude of the DTFT of $x_3[n]$, $|X_3(\omega)|$, over $-\pi < \omega < \pi$.

(ii) Express the new effective sampling rate, $F_s^{(3)}$, at the output in terms of the original sampling rate $F_s = 8W$?

(iii) Is there aliasing in the output? Yes or No. Briefly explain.

(iv) Is there loss of high frequency content in the output? Yes or No. Briefly explain.

(v) Express the output $x_3[n]$ in terms of a sampled version of the lowpass-filtered signal $x_{LP}(t)$ below, where $x_0(t)$ was defined in part (a). You can use $x_{LP}(t)$ in your answer to Problem 2(b.)

\[ x_{LP}(t) = x_0(t) \ast \frac{\sin\left(\frac{2\pi W}{3} t\right)}{\pi t} \]

- This effect a new sampling rate of $F_s^{(3)} = \frac{3}{2} F_s$.
- Since $F_s = 8W$ is Nyquist rate, this effects a sampling rate below the Nyquist rate.
- But the filter $h_3[n]$ removes high frequency content above $\frac{\pi}{2}$, thus, no aliasing.

- Filter gain of 3 offset by gain reduction by 3 due to decimation by 3.
Problem 2. This problem is about digital subbanding of the four DT signals \( x_i[n] \), \( i = 0, 1, 2, 3 \) from Problem 1. Digital subbanding of these four signals is effected in the efficient way via filter bank in Figure 1. All of the quantities in Figure 1 are defined below: the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

\[
h_{LP}[n] = \frac{\sin\left(\frac{\pi}{4} n\right)}{\pi n}
\]  
\( \quad (1) \)

The polyphase component filters on the left side of Figure 1 are defined as

\[
h_{\ell}^+[n] = h_{LP}[4n + \ell], \quad \ell = 0, 1, 2, 3.
\]  
\( \quad (2) \)

The respective signals at the inputs to these filters are formed from the input signals as

\[
\begin{bmatrix}
y_0[n] \\
y_1[n] \\
y_2[n] \\
y_3[n]
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
e^{-j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{4\pi}{4}} \\
e^{-j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{4\pi}{4}} \\
e^{-j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{4\pi}{4}}
\end{bmatrix}
\begin{bmatrix}
x_0[n] \\
x_1[n] \\
x_2[n] \\
x_3[n]
\end{bmatrix} \Rightarrow A =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
e^{-j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{4\pi}{4}} \\
e^{-j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{4\pi}{4}} \\
e^{-j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{4\pi}{4}}
\end{bmatrix}
\]  
\( \quad (3) \)

The polyphase component filters on the right side of Figure 1 are defined as

\[
h_{\ell}^-[n] = h_{LP}[4n - \ell], \quad \ell = 0, 1, 2, 3.
\]  
\( \quad (4) \)

The final output signals (on the far right side of Figure 1) are formed from linear combinations of the outputs of these filters via the matrix transformation below.

\[
\begin{bmatrix}
z_0[n] \\
z_1[n] \\
z_2[n] \\
z_3[n]
\end{bmatrix} =
\begin{bmatrix}
1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} \\
1 & 1 & 1 & 1 \\
e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} \\
e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}}
\end{bmatrix}
\begin{bmatrix}
s_0[n] \\
s_1[n] \\
s_2[n] \\
s_3[n]
\end{bmatrix} \Rightarrow B =
\begin{bmatrix}
1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi}{4}} \\
1 & 1 & 1 & 1 \\
e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} \\
e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi}{4}}
\end{bmatrix}
\]  
\( \quad (5) \)

\( \chi_0(n) \leftrightarrow \chi_{0}(\omega) \quad \text{centered at} \quad \omega = -\frac{\pi}{2} \)

\( \chi_1(n) \leftrightarrow \chi_{1}(\omega) \quad \text{at} \quad \omega = 0 \)

\( \chi_2(n) \leftrightarrow \chi_{2}(\omega) \quad \text{at} \quad \omega = \frac{\pi}{2} \)

\( \chi_3(n) \leftrightarrow \chi_{3}(\omega) \quad \text{at} \quad \omega = \pi \)
(i) $X_3(\omega)$

(ii) $F_5^{(3)} = \frac{2}{3} F_5 = \frac{2}{3} \cdot 8W = \frac{16W}{3}$

(iii) No aliasing

(iv) Yes, loss of high frequency content $\Rightarrow$ removed in order to avoid aliasing

(v) $X_{LP}(f)$

New sampling rate: $F_5^{(3)} = \frac{2}{3} 8W = \frac{16W}{3}$

$2W$ is mapped to $2\pi \frac{2W}{16W/3} = \frac{3\pi}{4}$

$\frac{8W}{3}$ is mapped to $2\pi \frac{\frac{8W}{3}}{16W/3} = \pi$ Same as $X_3(\omega)$ above

$\frac{1}{8W} \cdot \alpha = \frac{1}{16W/3} \Rightarrow \frac{3}{2}$

$X_3[n] = \frac{3}{2} X_{LP}(t) \bigg|_{t=n\frac{3}{16W}}$
Problem 2, part (a). Show all work. For all parts of this problem, \( h_{LP}[n] = 4 \frac{\sin \left( \frac{\pi}{4} n \right)}{\pi n} \).

(a)  
(i) Write a simple expression for the DTFT, \( H_2^-(\omega) \), of \( h_2^-[n] = h_{LP}[4n - 2] \) that holds for \(-\pi < \omega < \pi\).

(ii) Plot the phase \( \angle H_2^-(\omega) \) over \(-\pi < \omega < 3\pi\). Note that I am asking you to plot the phase, \( \angle H_2^-(\omega) \), from \(-\pi\) to \(3\pi\).
2(b) Express the output of the filter $h_2[n] = h_{LP}[4n + 2]$ in terms of sampled and time-shifted versions of the original analog input signal $x_0(t)$ and also $x_{LP}(t)$ defined in Prob. 1(d). You don’t need to write out the expressions for $x_0(t)$ and $x_{LP}(t)$. Keep in mind that all of the signals correspond to different sampling rates; make sure that’s clear in your answer. You don’t have to do a lot of work here but explain answers.

The 3rd row of $A$ is: 

\[
\begin{bmatrix}
-1 & 1 & -1
\end{bmatrix}
\]

\[
y_{[4n+2]} = -x_0 \left( nT_s^{(1)} + \frac{T_s^{(d)}}{2} \right) + x_0 \left( nT_s^{(1)} + \frac{T_s^{(1)}}{2} \right)
- x_0 \left( nT_s^{(2)} + \frac{T_s^{(2)}}{2} \right) + \frac{3}{2} x_{LP} \left( nT_s^{(3)} + \frac{T_s^{(3)}}{2} \right)
\]

\[
T_s^{(1)} = \frac{1}{8W} \quad T_s^{(1)} = \frac{1}{12W} \quad T_s^{(2)} = \frac{1}{16W} \quad T_s^{(3)} = \frac{3}{16W}
\]
2(c) The output of the filter $h_3[n] = h_{LP}[4n - 3]$ is denoted $s_3[n]$ in the block diagram. Express $s_3[n]$ in terms of $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$. You don’t need to write out the expressions for $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$. You don’t have to do a lot of work here; briefly explain your answer.

Since $h_3^+(n) = \delta(n)$, $s_3(n) = y_3(n)$

Last row of $A = \begin{bmatrix} 1 & -j & -1 \end{bmatrix}$

$s_3(n) = y_3(n) = j x_0(n) + x_1(n) - j x_2(n) - x_3(n)$
2(d) Plot the magnitude of the DTFT, $Y(\omega)$, of the interleaved signal $y[n]$ over $\pi < \omega < \pi$. Carefully label and graph the plot, clearly demarcating the subbands and showing which signal is in each subband. Clearly indicate the regions where $Y(\omega) = 0$. 

- everything is multiplied by 4
- This one should be centered at $\pi/2$
- This one should be centered at 0
2(e) Determine the convolution of \( h_1^+[n] = h_{LP}[4n + 1] \) with \( h_3^+[n] = h_{LP}[4n + 3] \), where \( h_{LP}[n] = 4 \frac{\sin \left( \frac{\pi}{4} n \right)}{\pi n} \). Simplify your answer as much as possible.

\[
g[n] = h_1^+[n] * h_3^+[n] = h_{LP}[4n + 1] * h_{LP}[4n + 3]
\]

Plot the phase, \( \angle G(\omega) \), of the DTFT of \( g[n] = h_1^+[n] * h_3^+[n] \). This problem is most easily solved via frequency domain analysis. You must show and explain your work.

\[
\overline{\omega} \quad -\pi < \omega < \pi:
\]

\[
G(\omega) = H_1^+(\omega) H_3^+(\omega) = e^{\frac{3\omega}{4}} e^{-\frac{3\omega}{4}}
\]

\[
= e^{\frac{3\omega}{4}}
\]

\[
\Rightarrow g(n) = \delta(n + 1)
\]

\[
\angle G(\omega)
\]
2(f) It is easy to show that $AB = 4I$ and $BA = 4I$, where $I$ is the 4x4 identity Matrix.
Plot the magnitude, $|Z_3(\omega)|$, of the DTFT of the output $z_3[n]$, over $-\pi < \omega < \pi$. 

\[ Z_3(n) = 4 \chi_3(n) \]