SOLUTION

NAME: 31 Oct. 2014
ECE 538 Digital Signal Processing I Exam 2 Fall 2014

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET
Test Duration: 60 minutes.
Open Book but Closed Notes.
One (both sides) handwritten 8.5 in x 11 in crib sheet allowed
Calculators NOT allowed.
All work should be done in the space provided.

There are two problems.
Problem 1 has 4 parts, 1(a) thru 1(d).
Problem 2 has 8 parts, 2(a) thru 2(h).

Continuous-Time Fourier Transform (Hz): $X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$

Continuous-Time Fourier Transform Pair (Hz): $\mathcal{F}\left\{\frac{\sin(2\pi Wt)}{\pi t}\right\} = rect\left\{\frac{F}{2W}\right\}$ where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = X_1(F) * X_2(F)$, where * denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(F)$, $i = 1, 2$.

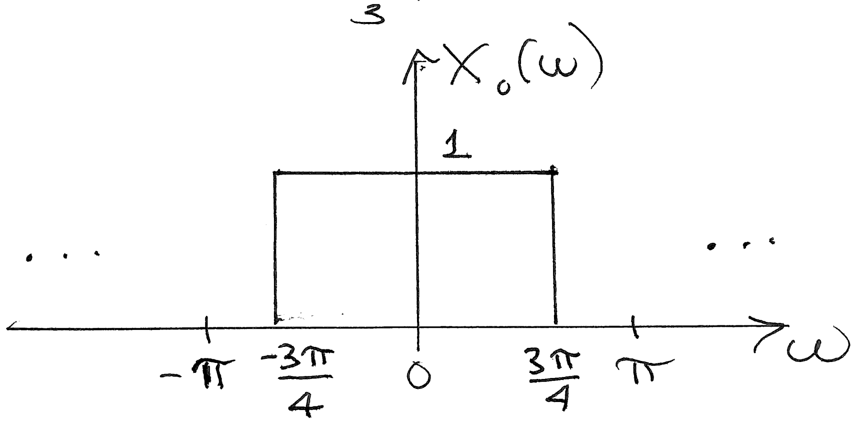
Relationship between DTFT and CTFT frequency variables in Hz: $\omega = 2\pi\frac{F}{F_s}$, where $F_s = \frac{1}{T_s}$ is the sampling rate in Hz

Prob. 1(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ with $F_s = \frac{1}{T_s} = \frac{8}{3}W$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_0(t) = T_s \frac{\sin(2\pi W t)}{\pi t}$$

max freq. in DT domain:

$$\omega_{max} = 2\pi \frac{W}{\frac{8}{3}W} = \frac{3}{4}\pi$$



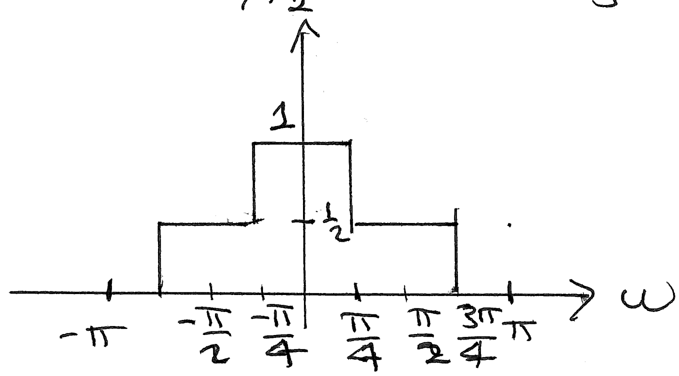
1(b) Consider the continuous-time signal $x_1(t)$ below. A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ with $F_s = \frac{1}{T_s} = \frac{8}{3}W$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_1(t) = T_s \frac{1}{2}\left\{\frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{3} t)}{\pi t}\right\}$$

from part (a)

$\frac{W}{3}$ mapped to

$X_1(\omega)$ $\quad 2\pi \frac{W}{3} \frac{1}{\frac{8}{3}W} = \frac{\pi}{4}$

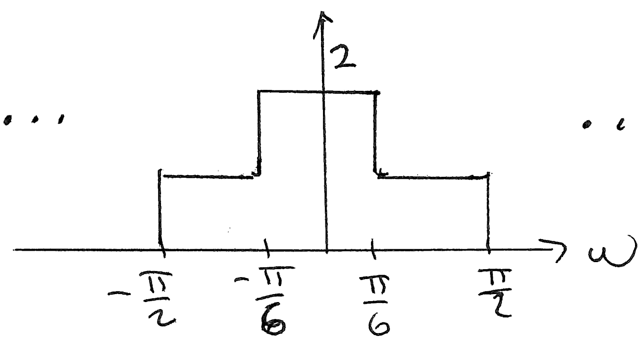


1(c) Consider the continuous-time signal $x_2(t)$ below. A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

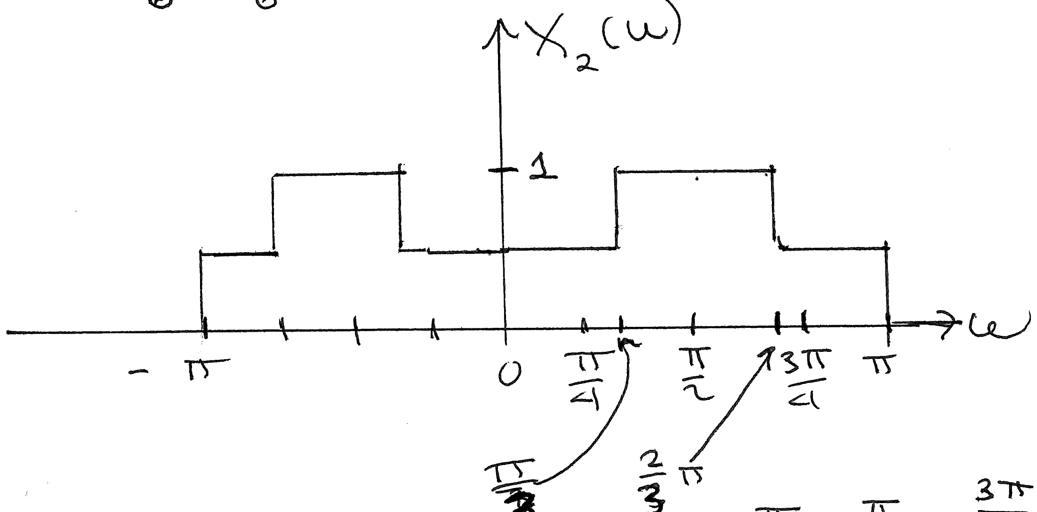
$$x_2(t) = T_s \left\{ \frac{\sin(2\pi W t)}{\pi t} + \frac{\sin(2\pi \frac{W}{3} t)}{\pi t} \right\} \cos(2\pi W t)$$

W gets mapped to $2\pi \frac{W}{4W} = \frac{\pi}{2}$

$\frac{W}{3}$ mapped to $2\pi \frac{W}{3} \frac{1}{4W} = \frac{\pi}{6}$



shifted to left and right by $\frac{\pi}{2}$ and divided by 2 in amplitude



$\frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6}$

$= \frac{2}{3}\pi$

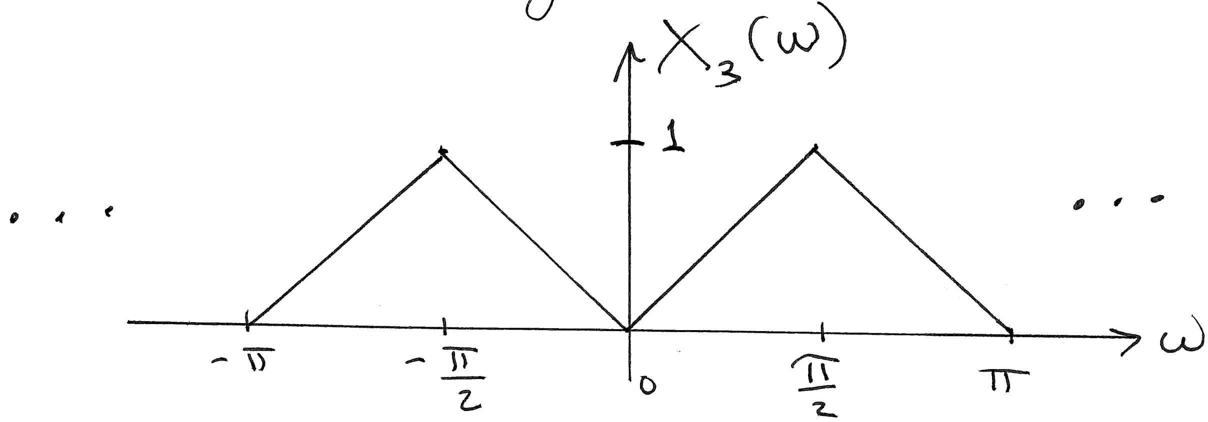
$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

1(d) Consider the continuous-time signal $x_3(t)$ below. A discrete-time signal is created by sampling $x_3(t)$ according to $x_3[n] = x_3(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_3[n]$, $|X_3(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_3(t) = T_s \frac{2}{W} \underbrace{\left\{ \frac{\sin\left(2\pi \frac{W}{2} t\right)}{\pi t} \right\}^2} \cos(2\pi W t)$$

because of squaring, max freq is doubled to W which is mapped to $2\pi \frac{W}{4W} = \frac{\pi}{2}$

so then the triangle is shifted to the left and right by $\pi/2$



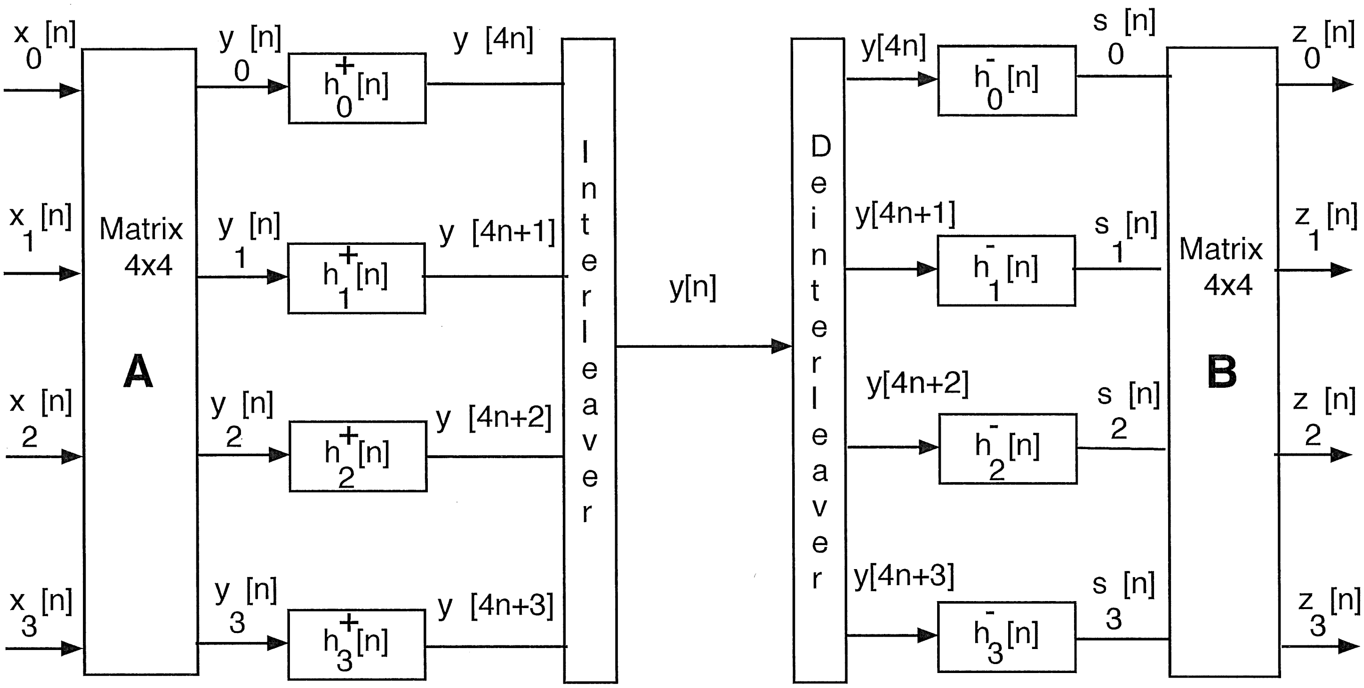


Figure 1.

Problem 2. This problem is about digital subbanding of the four DT signals $x_i[n]$, $i = 0, 1, 2, 3$ from Problem 1. Digital subbanding of these four signals is effected in the efficient way via filter bank in Figure 1. All of the quantities in Figure 1 are defined below: the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

$$h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 1 are defined as

$$h_\ell^+[n] = h_{LP}[4n + \ell], \quad \ell = 0, 1, 2, 3. \tag{2}$$

The respective signals at the inputs to these filters are formed from the input signals as

$$\begin{bmatrix} y_0[n] \\ y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-j\frac{2\pi}{4}} & 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} \\ e^{-j\frac{2\pi(2)}{4}} & 1 & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{4\pi(2)}{4}} \\ e^{-j\frac{2\pi(3)}{4}} & 1 & e^{j\frac{2\pi(3)}{4}} & e^{j\frac{4\pi(3)}{4}} \end{bmatrix} \begin{bmatrix} x_0[n] \\ x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-j\frac{2\pi}{4}} & 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} \\ e^{-j\frac{2\pi(2)}{4}} & 1 & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{4\pi(2)}{4}} \\ e^{-j\frac{2\pi(3)}{4}} & 1 & e^{j\frac{2\pi(3)}{4}} & e^{j\frac{4\pi(3)}{4}} \end{bmatrix} \tag{3}$$

The polyphase component filters on the right side of Figure 1 are defined as

$$h_\ell^-[n] = h_{LP}[4n - \ell], \quad \ell = 0, 1, 2, 3. \tag{4}$$

The final output signals (on the far right side of Figure 1) are formed from linear combinations of the outputs of these filters via the matrix transformation below.

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{2\pi(3)}{4}} \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi(2)}{4}} & e^{-j\frac{2\pi(3)}{4}} \\ 1 & e^{-j\frac{4\pi}{4}} & e^{-j\frac{4\pi(2)}{4}} & e^{-j\frac{4\pi(3)}{4}} \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{2\pi(3)}{4}} \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi(2)}{4}} & e^{-j\frac{2\pi(3)}{4}} \\ 1 & e^{-j\frac{4\pi}{4}} & e^{-j\frac{4\pi(2)}{4}} & e^{-j\frac{4\pi(3)}{4}} \end{bmatrix} \tag{5}$$

Problem 2, part (a). Show all work. For all parts of this problem, $h_{LP}[n] = 4\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$.

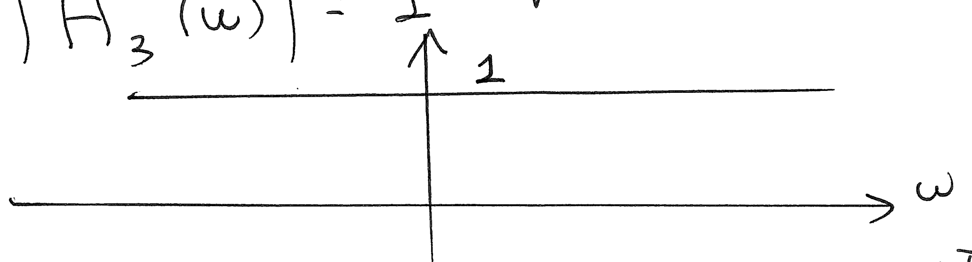
(a) (i) Determine and write a simplified expression for the DTFT, $H_3^-(\omega)$, of $h_3^-[n] = h_{LP}[4n-3]$ that holds for $-\pi < \omega < \pi$. Simplify as much as possible.

(ii) Plot the magnitude of $H_3^-(\omega)$ over $-\pi < \omega < \pi$.

(iii) Plot the phase $\angle H_3^-(\omega)$ over $-\pi < \omega < \pi$.

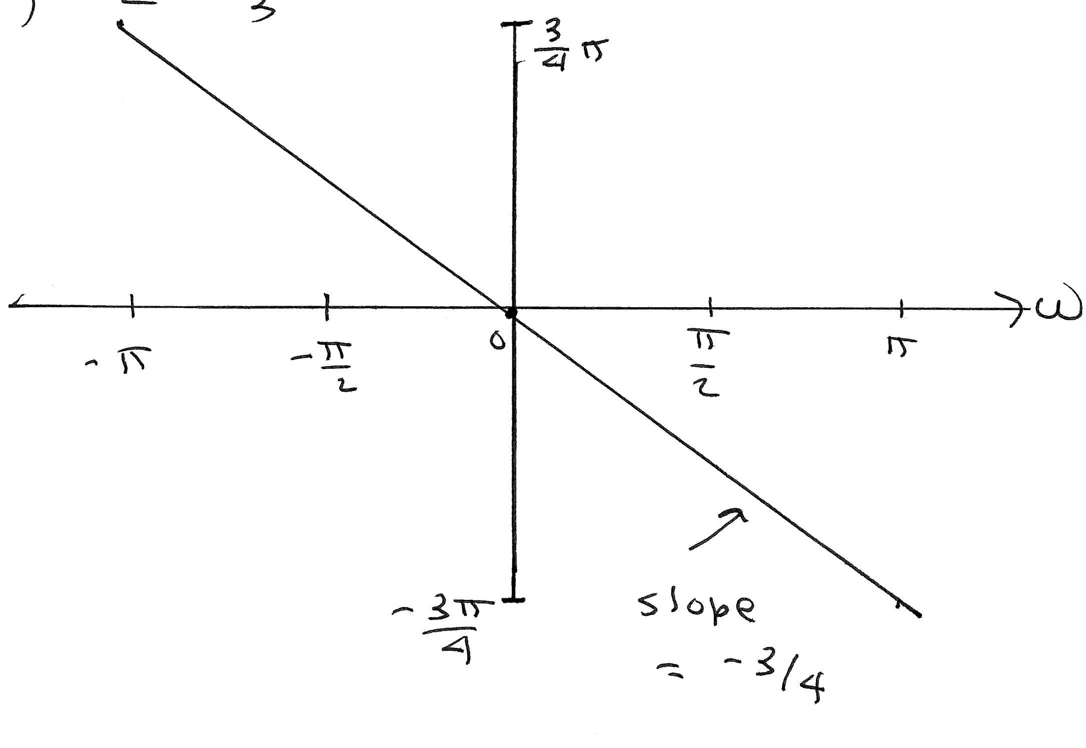
(a) $H_3^-(\omega) = e^{-j\frac{3}{4}\omega}$ for $-\pi < \omega < \pi$

(b) $|H_3^-(\omega)| = 1 \quad \forall \omega$



$-\pi < \omega < \pi$

(c) $\angle H_3^-(\omega) = -\frac{3}{4}\omega$



2(b) Express the output of the filter $h_2^+[n] = h_{LP}[4n+2]$ in terms of sampled and time-shifted versions of the original analog input signals $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$.

You don't need to write out the expressions for $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$. Also, just carry T_s along as a variable. You don't have to do a lot of work here; explain answer.

output of $h_2^+(n)$ is associated with 3rd row of A matrix which is simply $-1, 1, -1, 1$

and $h_2^+[n)$ effects a fractional time-shift of $\frac{2T_s}{4} = \frac{T_s}{2}$

$$y[4n+2) =$$

$$-x_0\left(nT_s^{(1)} + \frac{T_s^{(1)}}{2}\right) + x_1\left(nT_s^{(1)} + \frac{T_s^{(1)}}{2}\right) - x_2\left(nT_s^{(2)} + \frac{T_s^{(2)}}{2}\right) + x_3\left(nT_s^{(2)} + \frac{T_s^{(2)}}{2}\right)$$

It should be noted that T_s was different for the last two signals relative to the first two signals

$$T_s^{(1)} = \frac{1}{F_s^{(1)}} = \frac{1}{\frac{8}{3}W} = \frac{3}{8W}$$

$$T^{(2)} = \frac{1}{F_s^{(2)}} = \frac{1}{4W}$$

2(c) The output of the filter $h_1^-[n] = h_{LP}[4n-1]$ is denoted $s_1[n]$ in the block diagram. Express $s_1[n]$ in terms of sampled and possibly time-shifted versions of the original analog input signals $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$. You don't need to write out the expressions for $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$. Also, just carry T_s along as a variable. You don't have to do a lot of work here; briefly explain your answer.

$h_1^-[n]$ and $h_1^+[n]$ are effectively in series such that $h_1^+[n] * h_1^-[n] = \delta[n]$

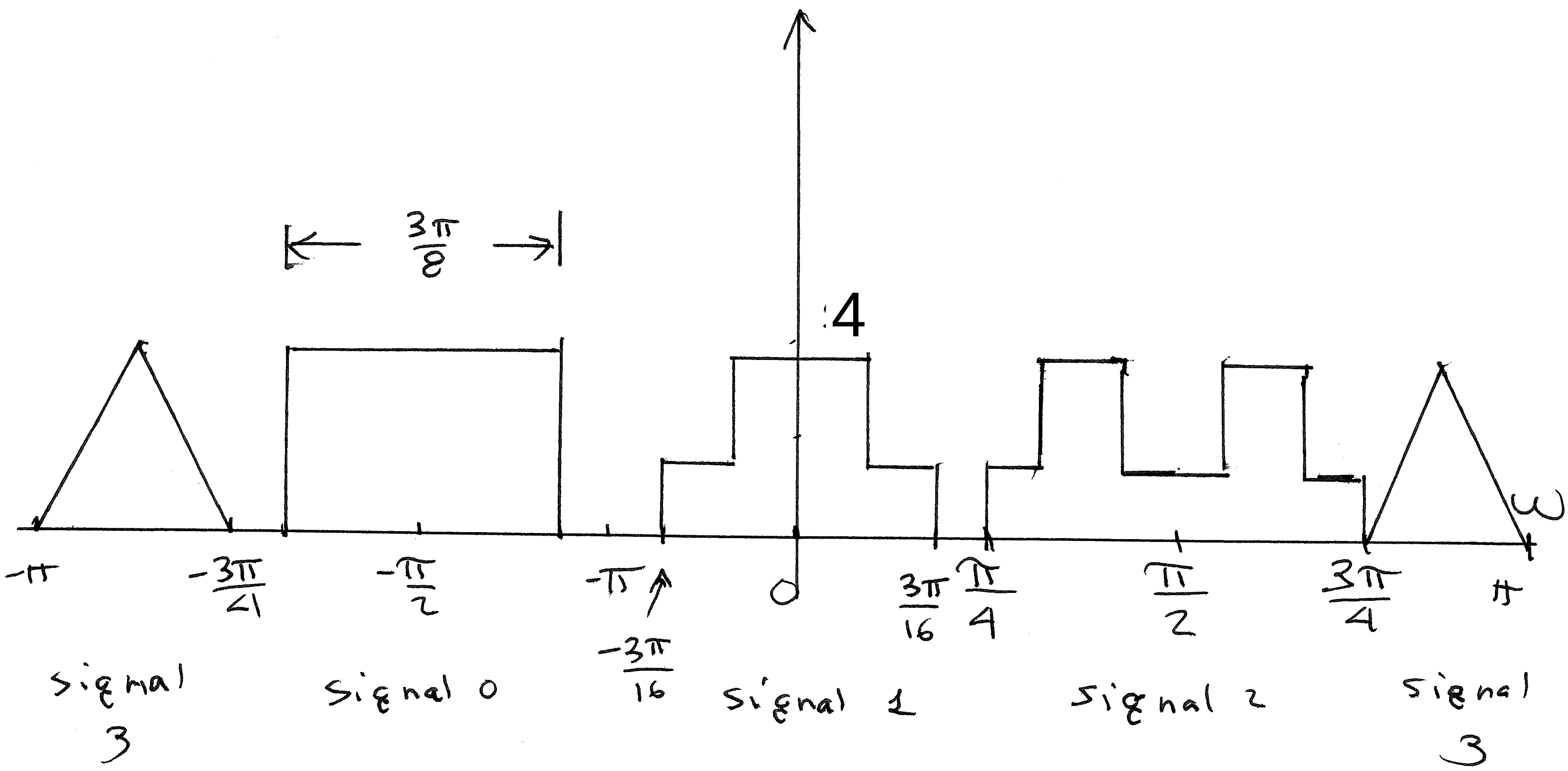
Thus, no time-shift so we just need the second row of the A matrix which is

$$\{-j, 1, j, -1\}$$

$$s_1[n] = -j\, x_0[n] + x_1[n] + j\, x_2[n] - x_3[n]$$

$$= -j\, x_0(nT_s^{(1)}) + x_1(nT_s^{(1)}) + j\, x_2(nT_s^{(2)}) - x_3(nT_s^{(2)})$$

2(d) Plot the magnitude of the DTFT, $Y(\omega)$, of the interleaved signal $y[n]$. Carefully label and graph the plot, clearly demarcating the subbands and showing which signal is in each subband.



2(e) Determine the convolution of $h_2^+[n] = h_{LP}[4n+2]$ with itself $h_2^+[n] = h_{LP}[4n+2]$, where $h_{LP}[n] = 4\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$. Simplify your answer for

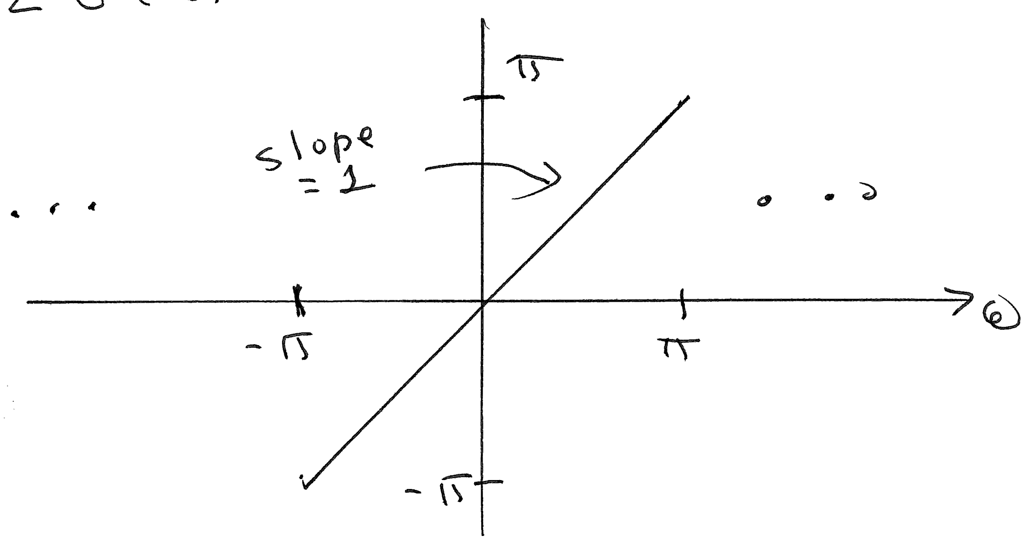
$$g[n] = h_2^+[n] * h_2^+[n] = h_{LP}[4n+2] * h_{LP}[4n+2] = ?$$

as much as possible. Plot the phase, $\angle G(\omega)$, of the DTFT of $g[n] = h_2^+[n] * h_2^+[n]$. This problem is most easily solved via frequency domain analysis. You must show and explain your work.

In the frequency domain

$$\left.\begin{aligned} G((\omega) &= H_2^+(\omega)\, H_2^+(\omega) \\ &= e^{j\frac{\omega}{2}}\, e^{j\frac{\omega}{2}} = e^{j\omega} \end{aligned}\right\} \text{Thus, } g(n) = \delta(n+1)$$

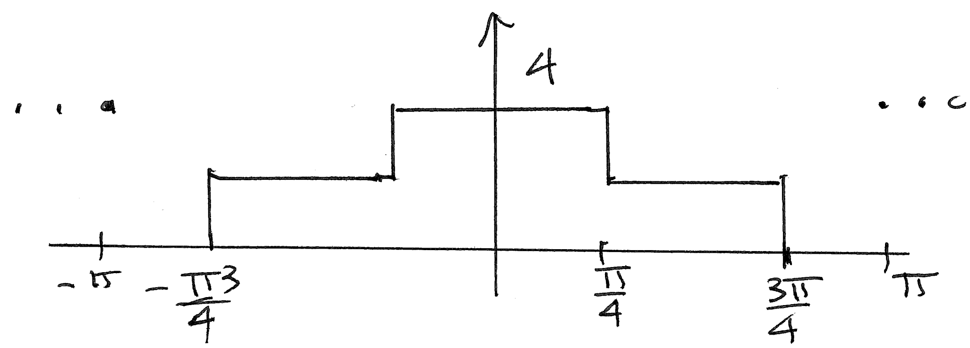
$\angle G(\omega) = \omega$ for $-\pi < \omega < \pi$



2(f) It is easy to show that $\mathbf{AB} = 4\mathbf{I}$ and $\mathbf{BA} = 4\mathbf{I}$, where $\mathbf{I}$ is the 4x4 identity Matrix. Plot the magnitude, $|Z_1(\omega)|$, of the DTFT of the output $z_1[n]$, over $-\pi < \omega < \pi$.

$$z_1[n] = 4\, x_1[n]$$

same as answer to 1(b) except for amplitude scaling by 4



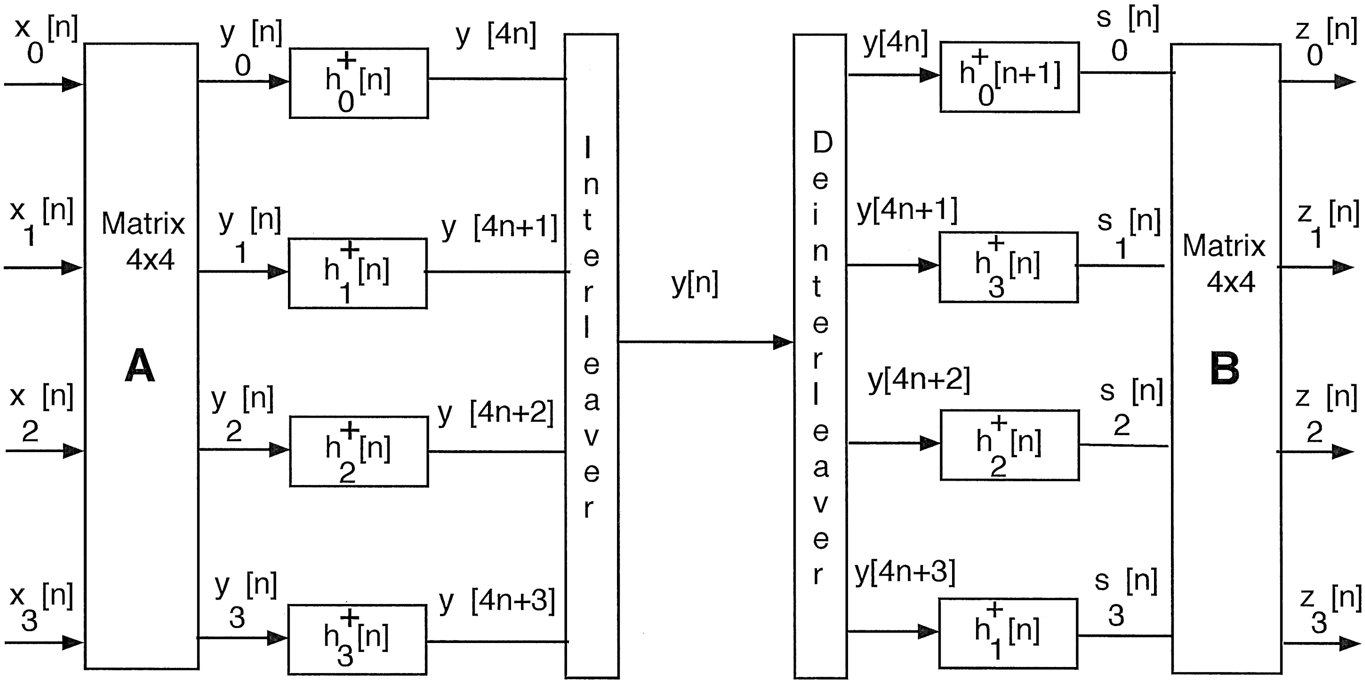


Figure 2.

2(g) Consider the system depicted in Figure 2 above. The 4x4 matrices **A** and **B** are as defined previously for the system in Figure 1, but there are some modifications relative to the filters on the right hand side. The output of the filter $h_3^+[n] = h_{LP}[4n+3]$ **on the right hand side** is denoted $s_1[n]$ in the block diagram. Express $s_1[n]$ in terms of sampled and possibly time-shifted versions of the original analog input signals $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$. (Part (h) on the next and final page, and refers to Fig 2. above.)

$$h_1^+[n] * h_3^+[n] = \delta[n+1]$$

and, as noted previously, 2nd row of A is simply $\{-j, 1, j, -1\}$

$$s_1[n] = -j\,x_0[n+1] + x_1[n+1] + j\,x_2[n+1] - x_3[n+1]$$

$$= -j\,x_0\left(nT_s^{(1)} + T_s^{(1)}\right) + x_1\left((n+1)T_s^{(1)}\right) + j\,x_2\left((n+1)T_s^{(2)}\right) - x_3\left((n+1)T_s^{(2)}\right)$$

2(h) It is easy to show that $\mathbf{AB} = 4\mathbf{I}$ and $\mathbf{BA} = 4\mathbf{I}$, where $\mathbf{I}$ is the 4x4 identity Matrix. For EACH output in Figure 2, express the output $z_k[n]$, in terms of $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$, for $k = 0, 1, 2, 3$. Explain your answers.

$$z_0[n] = 4\, x_0[n+1]$$

$$z_1[n] = 4\, x_1[n+1]$$

$$z_2[n] = 4\, x_2[n+1]$$

$$z_3[n] = 4\, x_3[n+1]$$

since:

$$h_0^+[n] * h_0^+[n+1]$$

$$= \delta[n] * \delta[n+1] = \delta[n+1]$$

$$h_1^+[n] * h_3^+[n] = \delta[n+1]$$

$$h_2^+[n] * h_2^+[n] = \delta[n+1]$$

$$h_3^*[n] * h_1^*[n] = \delta[n+1]$$

and deinterleaver is inverse system of interleaver