EXAM 2 DSP FALL 2001

P1)

a)

$$x(n) \longrightarrow \boxed{G(\omega)} \xrightarrow{z(n)} (M\uparrow) \xrightarrow{y(n)} \qquad \tilde{x}(n) \longrightarrow (M\uparrow) \xrightarrow{\tilde{z}(n)} \boxed{E(\omega)} \longrightarrow \tilde{y}(n)$$

$$Z(\omega) = X(\omega)\,G(\omega).$$

$$y(n) = \sum_{k=-\infty}^{\infty} z(k)\,\delta(n - kM)$$ (ie insert M-1 zeroes between each pair of $z(n)$).

$$\therefore \quad Y(\omega) = Z(M\omega)$$

Hence:

$$Y(\omega) = Z(M\omega) = \underline{X(M\omega)\,G(M\omega) = Y(\omega)}$$

also:

$$\tilde{Y}(\omega) = \tilde{Z}(\omega)\,E(\omega).$$

$$\tilde{z}(n) = \sum_{k=-\infty}^{\infty} \tilde{x}(k)\,\delta(n - kM)$$

$$\therefore \quad \tilde{Z}(\omega) = \tilde{X}(M\omega).$$

Hence:

$$\underline{\tilde{Y}(\omega) = \tilde{X}(M\omega)\,E(\omega)}$$

We want same I/O relation $\therefore$ $\underline{E(\omega) = G(M\omega)}$

b) $x(n) \longrightarrow \boxed{M\downarrow} \xrightarrow{z(n)} \boxed{H(\omega)} \xrightarrow{y(n)}$ $\qquad$ $\tilde{x}(n) \longrightarrow \boxed{F(\omega)} \xrightarrow{\tilde{z}(n)} \boxed{M\downarrow} \longrightarrow \tilde{y}(n)$

$$z(n) = x(nM) \quad \therefore \quad Z(\omega) = \frac{1}{M} \cdot \sum_{k=0}^{M-1} X\left(\frac{\omega - 2\pi k}{M}\right)$$

$$Y(\omega) = Z(\omega)\, H(\omega)$$

$$\therefore \quad Y(\omega) = \left(\frac{1}{M} \cdot \sum_{k=0}^{M-1} X\left(\frac{\omega - 2\pi k}{M}\right)\right) \cdot H(\omega)$$

$$\tilde{Z}(\omega) = \tilde{X}(\omega)\, F(\omega)$$

$$\tilde{y}(n) = \tilde{z}(nM) \quad \therefore \quad \tilde{Y}(\omega) = \frac{1}{M} \cdot \sum_{k=0}^{M-1} \tilde{Z}\left(\frac{\omega - 2\pi k}{M}\right)$$

Hence:

$$\tilde{Y}(\omega) = \frac{1}{M} \cdot \sum_{k=0}^{M-1} \tilde{X}\left(\frac{\omega - 2\pi k}{M}\right) \cdot F\left(\frac{\omega - 2\pi k}{M}\right)$$

$$\therefore \quad F\left(\frac{\omega - 2\pi k}{M}\right) = H(\omega) \quad \forall\, k = 0, 1, \ldots, M-1$$

ie:

$$F(\omega) = H(M\omega + 2\pi k) \qquad k = 0, \ldots, M-1$$

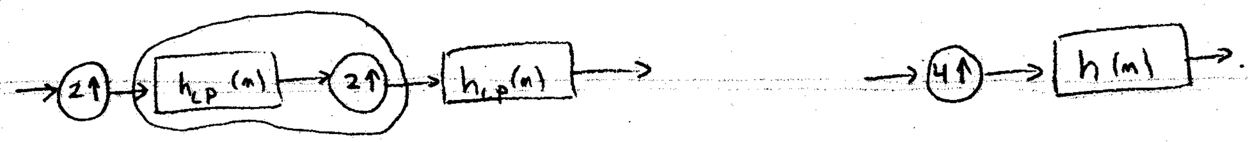
$H(M\omega)$ is periodic with period $2\pi/M$ $\therefore$ is periodic with period 2π and $\therefore$ $H(M\omega + 2\pi k) = H(M\omega)$ $\quad k = 0, \ldots, M-1$

$$\boxed{F(\omega) = H(M\omega)}$$

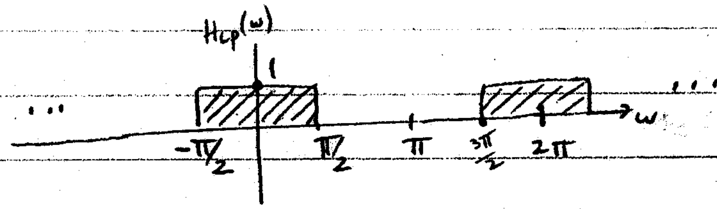
(Then: $F\left(\frac{\omega - 2\pi k}{M}\right) = H(\omega - 2\pi k) = H(\omega) \quad k = 0, 1, \ldots, M-1$)

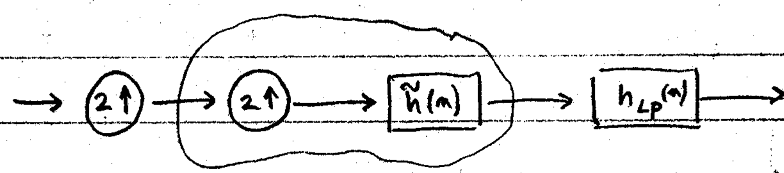
P2)

a)

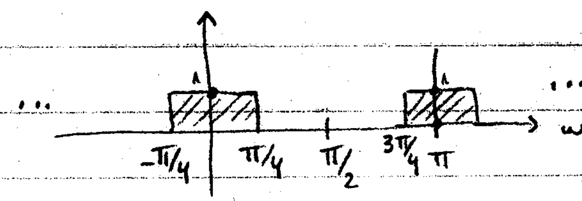


$$h_{LP}(n) = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$$





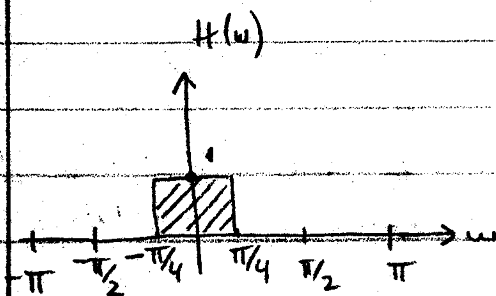
$$\tilde{H}(\omega) = H_{LP}(2\omega)$$

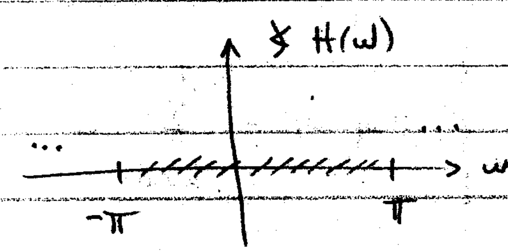


$\therefore$ $h(n) = h_{LP}(n) * \tilde{h}(n) \Rightarrow H(\omega) = H_{LP}(\omega)\,\tilde{H}(\omega)$

$$\therefore \quad \boxed{H(\omega) = H_{LP}(\omega)\, H_{LP}(2\omega)}$$

(both periodic 2π)





$$\therefore \quad h(n) = \frac{\sin(\frac{\pi}{4}n)}{\pi n}$$

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b)

$$h_1(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$$

$$H_1(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$$

$$\longrightarrow (2\uparrow) \longrightarrow (2\uparrow) \longrightarrow \boxed{\tilde{h}(n)} \longrightarrow \boxed{h_2(n)} \longrightarrow$$

$$\tilde{H}(\omega) = H_1(2\omega) = \underset{n=0}{1} + \underset{n=2}{e^{-j2\omega}} + \underset{n=4}{e^{-j4\omega}} + \underset{n=6}{e^{-j6\omega}}$$

$$\therefore \quad \tilde{h}(n) = \{\underset{\uparrow}{1}, 0, 1, 0, 1, 0, 1\}$$

$$h_{eq}(n) = \tilde{h}(n) * h_2(n)$$

$$= \{\underset{\uparrow}{1}, 0, 1, 0, 1, 0, 1\} * \{\underset{\uparrow}{1}, 1\}$$

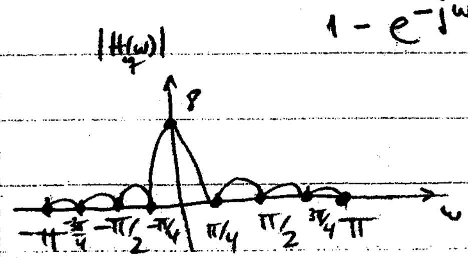
$$= \{\underset{\uparrow}{1}, 1, 1, 1, 1, 1, 1, 1\}$$

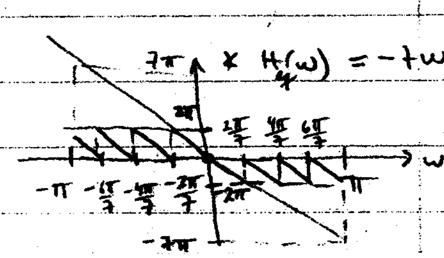
$$e^{+j8\omega} = 1 \Rightarrow 8\omega = 2\pi k, \quad \omega = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$$

$$H_{eq}(\omega) = \sum_{k=0}^{7} e^{-jk\omega} = \sum_{k=0}^{7} \left(e^{-j\omega}\right)^k \underset{(\omega \neq 0)}{=} \frac{1 - \left(e^{-j\omega}\right)^8}{1 - e^{-j\omega}}$$

$$= \frac{1 - e^{-j8\omega}}{1 - e^{-j\omega}}$$

(both period 2π)





c) zeros at $z = e^{\pm j\pi/2} = \pm j$

poles at

$$\frac{z-1}{z+1} = -1 \pm 2j$$

$$\Rightarrow \quad \cancel{z}\cancel{-1} = -z - 1 \pm 2j(z+1) = -\cancel{z}\cancel{-1} \pm 2jz \pm 2j$$

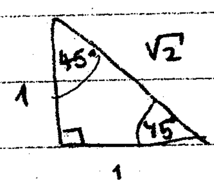
$$\Rightarrow \quad 2z \mp 2jz = \pm 2j$$

$$\Rightarrow \quad \cancel{2}z(1 \mp j) = \pm \cancel{2}j$$

$$\therefore \quad z = \frac{\pm j}{1 \mp j} = \begin{cases} \dfrac{j}{1-j} \\ \dfrac{-j}{1+j} \end{cases}$$

$$\frac{j}{1-j} = \frac{j(1+j)}{1-j^2} = \frac{j-1}{2}$$

$$= \frac{1}{2}\left(e^{j\pi/2} - 1\right) = \frac{1}{\cancel{2}} e^{j\pi/4}\underbrace{\left(e^{j\pi/4} - e^{-j\pi/4}\right)}_{\cancel{2}j\sin(\pi/4)}$$

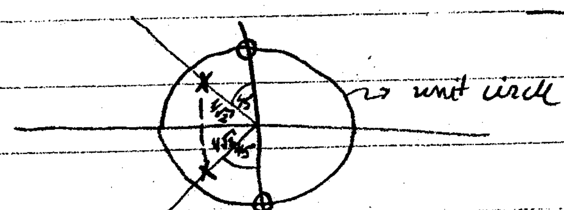


$$= j\, e^{j\pi/4} \sin(\pi/4)$$

$$= e^{j\left(\frac{\pi}{2} + \frac{\pi}{4}\right)} \sin(\pi/4)$$

$$\sin(\pi/4) = \sin(45^\circ) = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \text{poles at } z = \frac{1}{\sqrt{2}} e^{\pm j\left(\frac{\pi}{2} + \frac{\pi}{4}\right)}$$



d) $$H(s) = K \cdot \frac{(s-j)(s+j)}{\big(s-(-1-2j)\big)\big(s-(-1+2j)\big)} = K \cdot \frac{s^2 - j^2}{s^2+2s+5}$$

$$= K \cdot \frac{s^2+1}{s^2+2s+5}$$

$\therefore$

$$H(z) = K \cdot \frac{\left(\frac{z-1}{z+1}\right)^2 + 1}{\left(\frac{z-1}{z+1}\right)^2 + 2\left(\frac{z-1}{z+1}\right) + 5} = K \cdot \frac{2(z^2+1)}{8\left(z^2+z+\frac{1}{2}\right)}$$

$$= \frac{K}{4} \cdot \frac{(z^2+1)}{\left(z^2+z+\frac{1}{2}\right)}$$

$$H(\omega=0) = 0.8 \Rightarrow H(z=1) = 0.8 = \frac{K}{\cancel{4}} \cdot \frac{\cancel{2}}{5/\cancel{2}}$$

$$\therefore \quad K = 5 \cdot 0.8 = \underline{4}$$

$$\therefore \quad H(z) = \boxed{\frac{z^2+1}{z^2+z+1/2}}$$

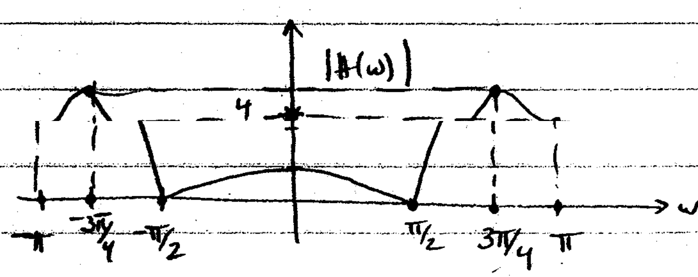
$$z=-1 \ (\omega=\pi) \Rightarrow H(-1) = \frac{2}{1-1+1/2} = 4$$

$\therefore$ $H(\omega=0) = 0.8$

$H(\omega=\pm\pi/2) = 0$ (zeroes at $\pm j$)

$H(\omega=\pm\pi) = 4$

Poles at $\omega = \pm 3\pi/4$



P3) $p_{s,1} = -1+2j$, $p_{s,2} = -1-2j$ } is stable since $Re(p_i)<0$ 7

$z_{s,1} = j$

$z_{s,2} = -j$

$$s = \frac{z-1}{z+1}$$

a) Yes, because the analog filter is stable
∴ the digital filter obtained via the bilinear transformation is BIBO stable //
(LHP of s-plane is mapped into inside of unit circle).

b) $\omega = 2\,\mathrm{Arctg}\left(\Omega/c\right)$ $\quad c=1$

$= 2\,\mathrm{Arctg}(\Omega)$

$s = \sigma + j\Omega$ $\quad$ zeros at $s = \pm j$ $\quad \therefore \; \Omega = \pm 1$

$$\therefore \quad \omega = 2\,\mathrm{Arctg}(\pm 1) = \pm 2 \cdot \underbrace{\mathrm{Arctg}(1)}_{\pi/4}$$

$$= \pm \pi/2$$

$$\therefore \quad \boxed{\omega = \pi/2}$$

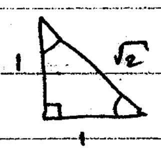
c) $H(z) = \dfrac{z^2+1}{z^2+z+1/2} = \dfrac{1+z^{-2}}{1+z^{-1}+\frac{1}{2}z^{-2}} = \dfrac{Y(z)}{X(z)}$

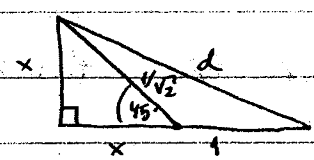
$\therefore \quad Y(z)\left(1+z^{-1}+\frac{1}{2}z^{-2}\right) = X(z)\left(1+z^{-2}\right)$

ie:

$$y(n) + y(n-1) + \frac{1}{2}y(n-2) = x(n) + x(n-2)$$

(note: d) can also be done as follows:





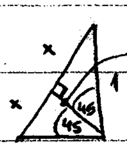
$2x^2 = 1/2$

$\Rightarrow \quad x = 1/2$

$$d^2 = \left(1+1/2\right)^2 + \left(1/2\right)^2 \quad \Rightarrow \quad d = \sqrt{5}/\sqrt{2}$$

$$\therefore \quad H(0) = G \cdot \frac{(\sqrt{2})^2}{(\sqrt{5}/\sqrt{2})^2} = G \cdot \frac{2}{5/2} = G \cdot \frac{4}{5} = 0.8\,G \overset{!}{=} 0.8$$

$\therefore \quad \underline{G = 1}$



$\cos 45^\circ = 1/\sqrt{2}$

$$\therefore \quad |H(\pi)| = G \cdot \frac{(2x)^2}{x^2} = 4$$

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