NAME: SOLUTION 1 Nov. 2013

ECE 538 Digital Signal Processing I Exam 2 Fall 2013

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET

Test Duration: 60 minutes.

Open Book but Closed Notes.

Calculators NOT allowed.

All work should be done in the space provided.

Continuous-Time Fourier Transform (rads/sec): $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Continuous-Time Fourier Transform Pair (rads/sec): $\mathcal{F}\left\{\frac{\sin(Wt)}{\pi t}\right\} = rect\left\{\frac{\omega}{2W}\right\}$
where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$, where $*$ denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(\omega)$, $i = 1, 2$.

Relationship between DTFT and CTFT frequency variables in rads/sec: $\omega = \Omega T_s$

Relationship between DTFT and CTFT frequency variables in Hz: $\omega = 2\pi\frac{F}{F_s}$, where $F_s = \frac{1}{T_s}$ is the sampling rate in Hz

Problem 1 (a). Consider an analog signal with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 60$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{60}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{\frac{60}{2\pi}\right\}^2 \frac{\sin(\frac{\pi}{6}n)}{\pi n}\frac{\sin(\frac{\pi}{2}n)}{\pi n} \qquad \text{where: } \; T_s = \frac{2\pi}{60}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h\left(t - nT_s\right) \qquad \text{where: } \; T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s\frac{\pi}{10}\frac{\sin(10t)}{\pi t}\frac{\sin(30t)}{\pi t}$$

Problem 1 (b). Consider the SAME analog signal with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{60}$ sec, but at a different starting point. This yields the Discrete-Time $x[n]$ signal below, where $0 < \epsilon < 1$.

$$x_\epsilon[n] = x_a(nT_s + \epsilon T_s) = \left\{\frac{60}{2\pi}\right\}^2 \frac{\sin(\frac{\pi}{6}(n+\epsilon))}{\pi(n+\epsilon)}\frac{\sin(\frac{\pi}{2}(n+\epsilon))}{\pi(n+\epsilon)} \qquad \text{where: } \; T_s = \frac{2\pi}{60}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Does your final answer depend on the value of ϵ? Explain your answer.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n]h\left(t - (n+\epsilon)T_s\right) \qquad \text{where: } \; T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s\frac{\pi}{10}\frac{\sin(10t)}{\pi t}\frac{\sin(30t)}{\pi t}$$

1 (a) $\boxed{\text{Solved in rads/sec} \Rightarrow \text{you can divide frequencies by } 2\pi \text{ to solve in Hz, if you like}}$

$\omega_M = 20 \Rightarrow \omega_{Nyquist} = 40 \Rightarrow \omega_s = 60 > 2\omega_M$

$\Rightarrow$ no aliasing!

- First replica centered at $\omega_s = 60 \Rightarrow$ lower edge at $\omega_s - \omega_M = 40$
- Thus, don't care region from $20 < \omega < 40$
- The $h(t)$ given is flat over $|\omega| < 20$, then rolls-off linearly down to zero over $20 < \omega < 40$
- End result: perfect reconstruction:

$$\boxed{x_r(t) = x(t) = x[n]} \quad n = \frac{t}{T_s} = t\,\frac{60}{2\pi}$$

$$= \left\{\frac{60}{2\pi}\right\}^2 \frac{\sin\left(\frac{\pi}{6}\frac{60}{2\pi}t\right)\sin\left(\frac{\pi}{2}\frac{60}{2\pi}t\right)}{\pi^2\left(t\,\frac{60}{2\pi}\right)^2}$$

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$$x_r(t) = \frac{\sin(5t)}{\pi t}\,\frac{\sin(15t)}{\pi t} \Leftarrow \text{answer}$$

Prob. 1 (b)

Same answer as (a).

Since sampling above Nyquist rate, it doesn't what point you start sampling at $\Rightarrow$ you just have to center the interpolating fns. at the time points where you sampled

Problem 2. GIVEN: Each of the four signals in this problem has the same bandwidth (i.e., same maximum frequency) and is sampled at the same rate which is ABOVE (greater than) the Nyquist rate (no aliasing!)

(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ for $T_s = \frac{3\pi}{40}$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

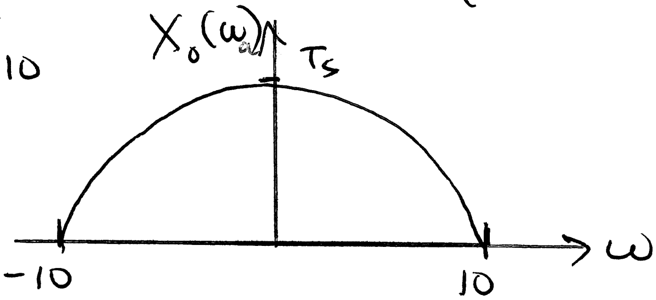
$$x_0(t) = T_s \frac{1}{2}\left\{ \frac{\sin(10(t-\frac{\pi}{20}))}{\pi(t-\frac{\pi}{20})} + \frac{\sin(10(t+\frac{\pi}{20}))}{\pi(t+\frac{\pi}{20})} \right\}$$

Use CTFT-DTFT relationship:

CTFT: $X_0(\omega_a) = T_s \underbrace{\text{rect}\left(\frac{\omega}{20}\right)}_{\text{see top of page 2}} \frac{1}{2}\left\{ e^{-j\frac{\pi}{20}\omega} + e^{+j\frac{\pi}{20}\omega} \right\}$

$= T_s \cos\left(\frac{\pi}{20}\omega\right) \text{rect}\left(\frac{\omega}{20}\right)$

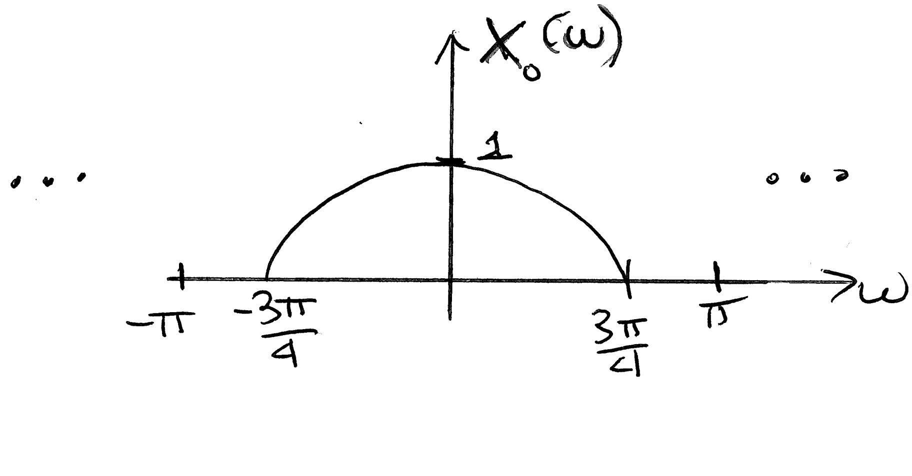
$= 1$ for $|\omega| < 10$



$\omega_d = \frac{\omega_a}{F_s} = \omega_a T_s$

$\omega_M = 10$ mapped to $10 \cdot T_s$

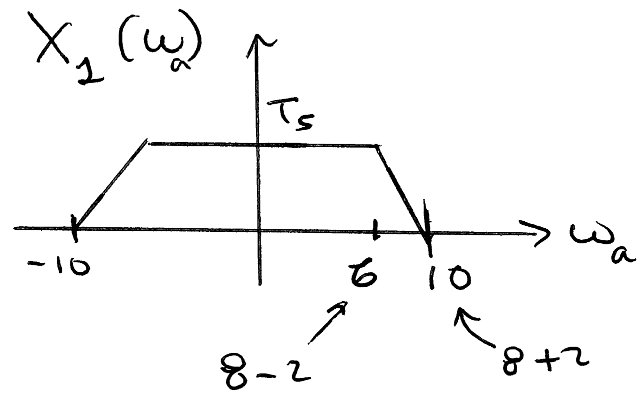
$= 10 \cdot \frac{3\pi}{40} = \frac{3\pi}{4}$



(b) Consider the continuous-time signal $x_1(t)$ below. A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ for $T_s = \frac{3\pi}{40}$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_1(t) = T_s \cancel{\frac{1}{2\pi}} \left\{ \frac{\sin(2t)}{\pi t} \, \frac{\sin(8t)}{\pi t} \right\} \frac{\pi}{2}$$

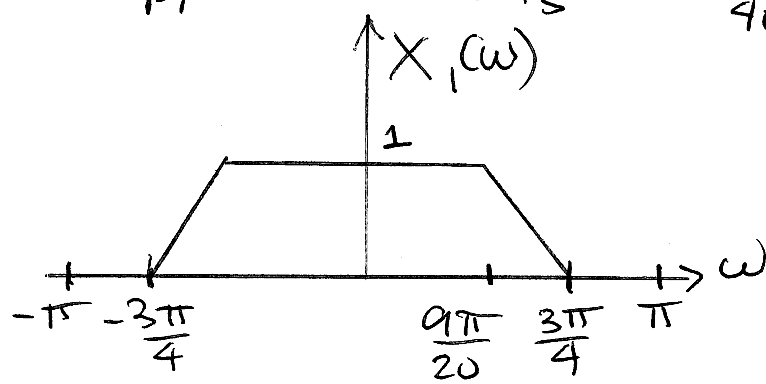
CTFT:



CTFT-DTFT relationship: $\omega_d = \omega_a T_s$

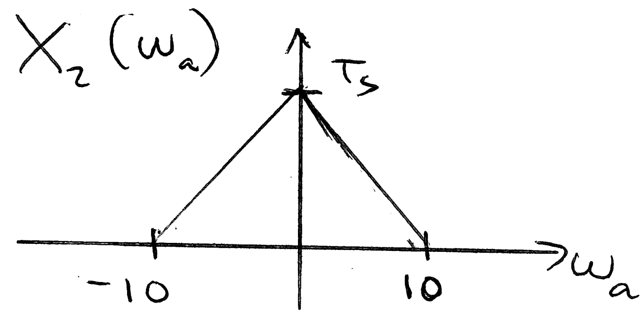
$\omega = 8$ mapped to $6T_s = 6\,\frac{3\pi}{40} = \frac{9\pi}{20}$

$\omega = 10$ mapped to $10T_s = 10\,\frac{3\pi}{40} = \frac{3}{4}\pi$



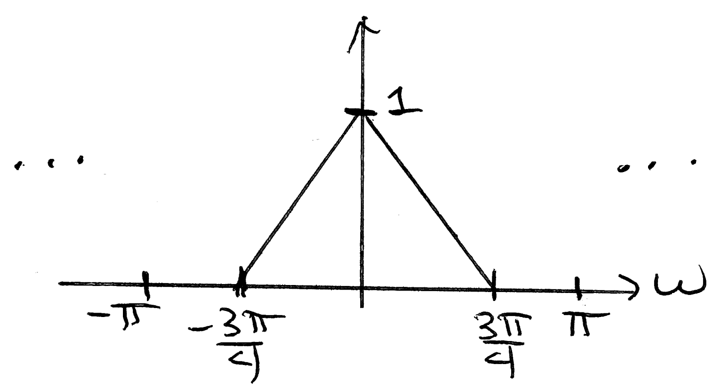
(c) Consider the continuous-time signal $x_2(t)$ below. A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2(nT_s)$ for $T_s = \frac{3\pi}{40}$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_2(t) = T_s \frac{\pi}{5} \left\{ \frac{\sin(5t)}{\pi t} \right\}^2$$



Use CTFT-DTFT relationship:

10 mapped to $10T_s = \frac{3\pi}{4}$ just like the other parts



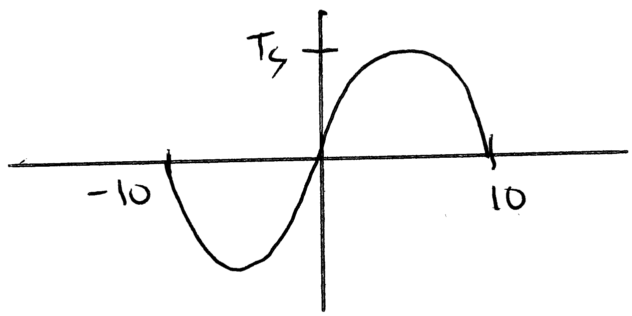
(d) Consider the continuous-time signal $x_3(t)$ below. A discrete-time signal is created by sampling $x_3(t)$ according to $x_3[n] = x_3(nT_s)$ for $T_s = \frac{3\pi}{40}$. Plot the magnitude of the DTFT of $x_3[n]$, $|X_3(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_3(t) = T_s \frac{1}{2j} \left\{ \frac{\sin(10(t - \frac{\pi}{10}))}{\pi(t - \frac{\pi}{10})} - \frac{\sin(10(t + \frac{\pi}{10}))}{\pi(t + \frac{\pi}{10})} \right\}$$

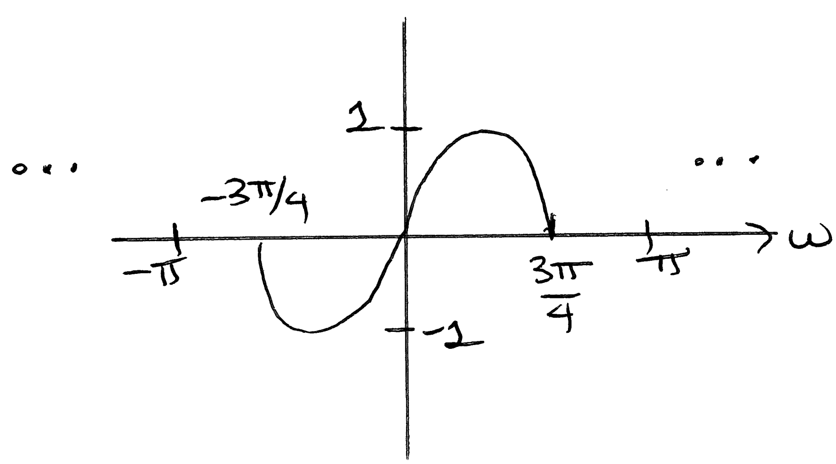
CTFT:

$$X_3(\omega_a) = T_s \operatorname{rect}\left(\frac{\omega}{20}\right) \frac{1}{2j} \left\{ e^{-j\frac{\pi}{10}\omega} - e^{j\frac{\pi}{10}\omega} \right\}$$

$$= -T_s \sin\left(\frac{\pi}{10}\omega\right) \operatorname{rect}\left(\frac{\omega}{20}\right)$$



CTFT-DTFT relationship:



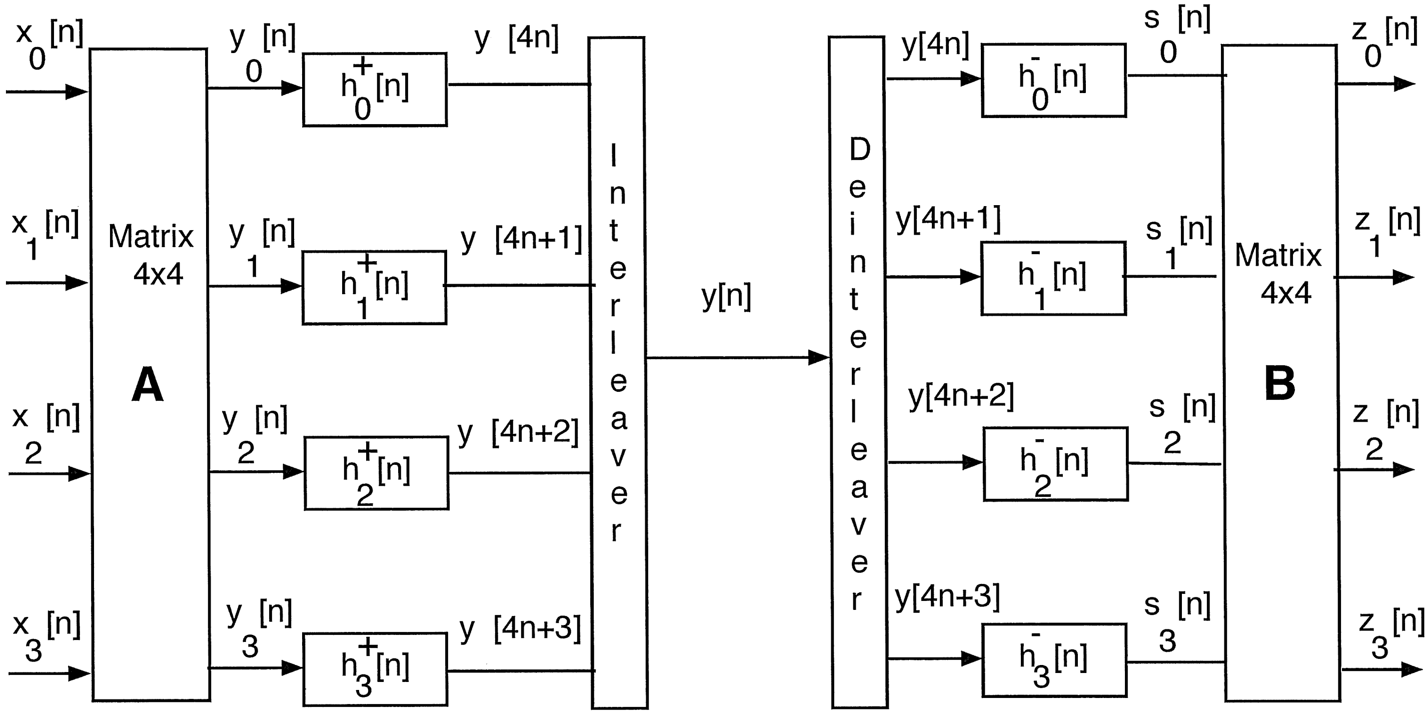


Figure 1.

Problem 3. This problem is about digital subbanding of the four DT signals $x_i[n]$, $i = 0, 1, 2, 3$ from Problem 2. Digital subbanding of these four signals is effected in the efficient way via filter bank in Figure 1. All of the quantities in Figure 1 are defined below: the respective impulse responses of the polyphase component filters are defined in terms of the ideal lowpass filter impulse response below.

$$h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 1 are defined as

$$h_\ell^+[n] = h_{LP}[4n+\ell], \quad \ell = 0, 1, 2, 3. \tag{2}$$

The respective signals at the inputs to these filters are formed from the input signals as

$$\begin{bmatrix} y_0[n] \\ y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-j\frac{2\pi}{4}} & 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} \\ e^{-j\frac{2\pi(2)}{4}} & 1 & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{4\pi(2)}{4}} \\ e^{-j\frac{2\pi(3)}{4}} & 1 & e^{j\frac{2\pi(3)}{4}} & e^{j\frac{4\pi(3)}{4}} \end{bmatrix} \begin{bmatrix} x_0[n] \\ x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{-j\frac{2\pi}{4}} & 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} \\ e^{-j\frac{2\pi(2)}{4}} & 1 & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{4\pi(2)}{4}} \\ e^{-j\frac{2\pi(3)}{4}} & 1 & e^{j\frac{2\pi(3)}{4}} & e^{j\frac{4\pi(3)}{4}} \end{bmatrix} \tag{3}$$

The polyphase component filters on the right side of Figure 1 are defined as

$$h_\ell^-[n] = h_{LP}[4n-\ell], \quad \ell = 0, 1, 2, 3. \tag{4}$$

The final output signals (on the far right side of Figure 1) are formed from linear combinations of the outputs of these filters via the matrix transformation below.

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{2\pi(3)}{4}} \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi(2)}{4}} & e^{-j\frac{2\pi(3)}{4}} \\ 1 & e^{-j\frac{4\pi}{4}} & e^{-j\frac{4\pi(2)}{4}} & e^{-j\frac{4\pi(3)}{4}} \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} \Rightarrow \mathbf{B} = \begin{bmatrix} 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{2\pi(2)}{4}} & e^{j\frac{2\pi(3)}{4}} \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j\frac{2\pi(2)}{4}} & e^{-j\frac{2\pi(3)}{4}} \\ 1 & e^{-j\frac{4\pi}{4}} & e^{-j\frac{4\pi(2)}{4}} & e^{-j\frac{4\pi(3)}{4}} \end{bmatrix} \tag{5}$$

Problem 3 part (a). Show all work. For all parts of this problem, $h_{LP}[n] = 4\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$.

(a) (i) Determine and write a simplified expression for the DTFT, $H_3^+(\omega)$, of $h_3^+[n] = h_{LP}[4n+3]$ that holds for $-\pi < \omega < \pi$. Simplify as much as possible.

(ii) Plot the magnitude of $H_3^+(\omega)$ over $-\pi < \omega < \pi$.

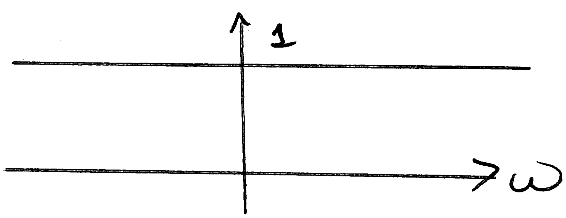
(iii) Plot the phase of $H_3^+(\omega)$ over $-\pi < \omega < \pi$.

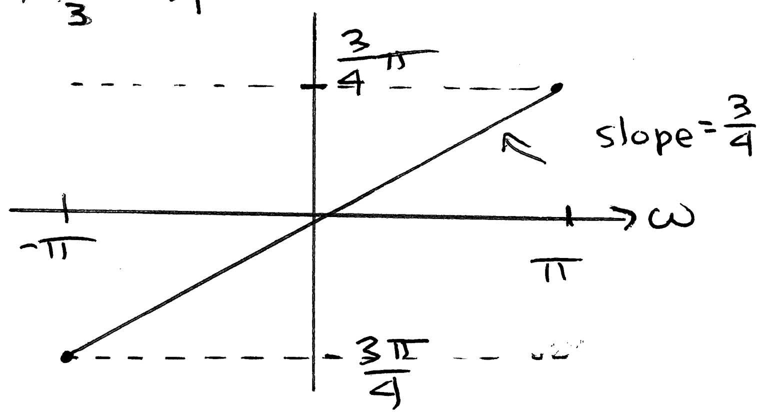
Since ideal case: $H_3^+(\omega) = e^{j\frac{3}{4}\omega}$

for $-\pi < \omega < \pi$

$|H_3^+(\omega)| = 1$

$\angle H_3^+(\omega) = \frac{3}{4}\omega$ over $-\pi < \omega < \pi$





(b) Express the output of the filter $h_3^+[n] = h_{LP}[4n+3]$ in terms of sampled and time-shifted versions of the original analog input signals $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$.

You don't need to write out the expressions for $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$. Also, just carry T_s along as a variable, rather than having to write its specific numerical value of $T_s = \frac{3\pi}{40}$. You don't have to do a lot of work here; briefly explain your answer.

$$y[4n+3] = [\text{last row of } A] \begin{bmatrix} x_0(t+.75T_s) \\ x_1(t+.75T_s) \\ x_2(t+.75T_s) \\ x_3(t+.75T_s) \end{bmatrix}$$

$$= e^{-j\frac{2\pi(3)}{4}} x_0\left(t+\tfrac{3}{4}T_s\right) + x_1\left(t+\tfrac{3}{4}T_s\right) + e^{j\frac{2\pi(2)}{4}} x_2\left(t+\tfrac{3}{4}T_s\right)$$
$$+ e^{j\frac{4\pi(3)}{4}} x_3\left(t+\tfrac{3}{4}T_s\right)$$

this follows from answer to 3(a)

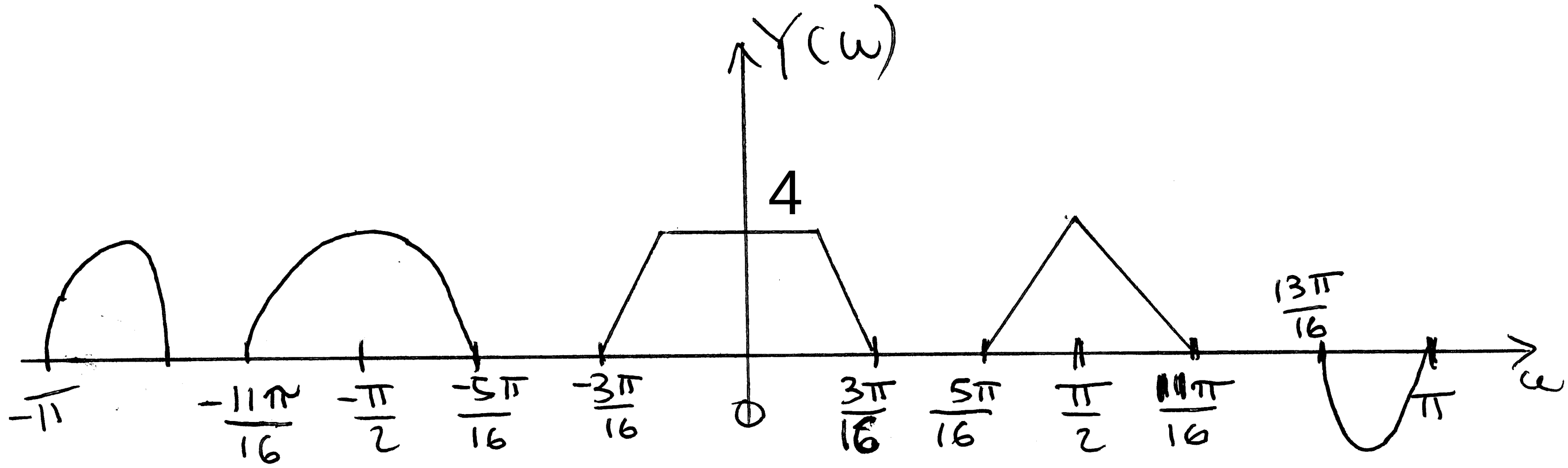
(c) Plot the magnitude of the DTFT, $Y(\omega)$, of the interleaved signal $y[n]$. Carefully label and graph the plot, clearly demarcating the subbands and showing which signal is in each subband.

$x_0[n]$ centered at $-\frac{2\pi}{4} = -\frac{\pi}{2}$

$x_1[n]$ centered at $\frac{2\pi}{4} = \frac{\pi}{2}$

$x_2[n]$ centered at $\frac{2\pi}{4} = \frac{\pi}{2}$

$x_3[n]$ centered at $\frac{4\pi}{4} = \pi$



$\frac{\pi}{2} - \frac{3\pi}{16}$

$= \frac{8\pi}{16} - \frac{3\pi}{16} = \frac{5\pi}{16}$

(d) Determine the convolution of $h_1^+[n] = h_{LP}[4n+1]$ and $h_3^+[n] = h_{LP}[4n+3]$, where $h_{LP}[n] = 4\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$.

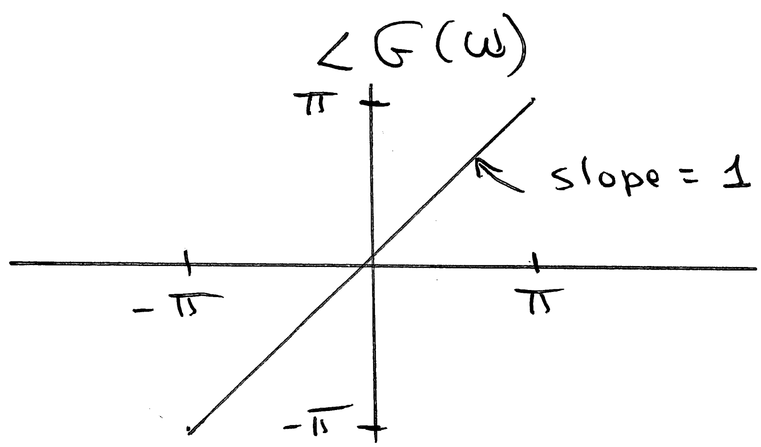
$$g[n] = h_1^+[n] * h_3^+[n] = h_{LP}[4n+1] * h_{LP}[4n+3] = ?$$

Plot the phase, $\angle G(\omega)$, of the DTFT of $g[n] = h_1^+[n] * h_3^+[n]$. This problem is most easily solved via frequency domain analysis. You must show and explain your work.

$$h_\ell^*[n] \xleftrightarrow{\text{DTFT}} H_\ell^+(\omega) = e^{j\frac{\ell}{4}\omega} \qquad -\pi < \omega < \pi$$

$$h_1^*[n] * h_3^+[n] \xleftrightarrow{\text{DTFT}} e^{j\frac{\omega}{4}} e^{j\frac{3}{4}\omega} = e^{j\omega}$$

Thus, $g[n] = \delta[n+1]$



(e) It is easy to show that $\mathbf{AB} = 4\mathbf{I}$ and $\mathbf{BA} = 4\mathbf{I}$, where $\mathbf{I}$ is the 4x4 identity Matrix. For EACH output: express the output $z_k[n]$, in terms of $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$, for $k = 0, 1, 2, 3$. Explain your answers.

$z_k[n] = 4\, x_k[n]$, $k = 0, 1, 2, 3$

Since $h_l^+[n] * h_l^-[n] = \delta[n]$

and $AB = BA = 4I$

and interleaver and deinterleaver are inverse operations

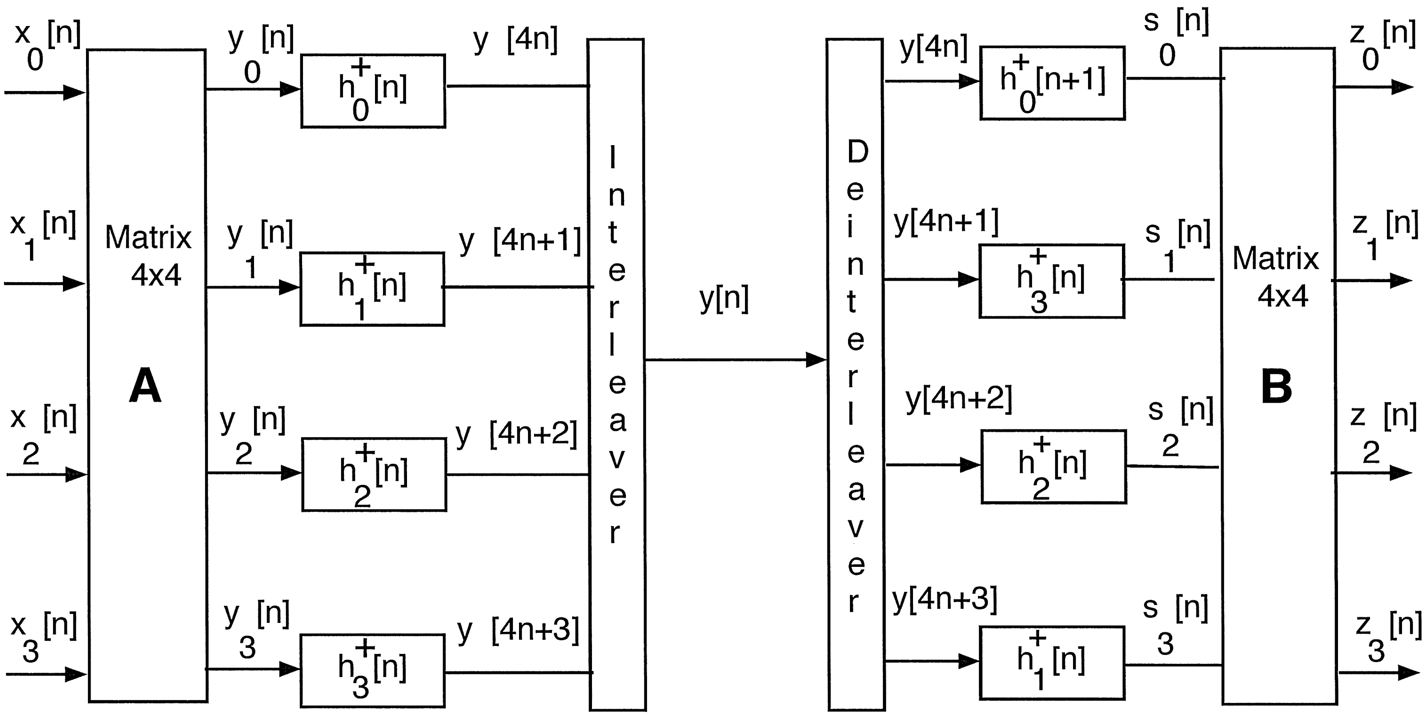


Figure 2.

(f) Analyze the system depicted in Figure 2 above. Look carefully: there are some modifications relative to to the system in Figure 1, particularly the filters used on the right hand side. The 4x4 matrices **A** and **B** are the same as those defined previously for the system in Figure 1, satisfying $\mathbf{AB} = 4\mathbf{I}$, where $\mathbf{I}$ is the 4x4 identity Matrix. For EACH output: express the output $z_k[n]$, in terms of $x_0[n]$, $x_1[n]$, $x_2[n]$, and $x_3[n]$, for $k = 0, 1, 2, 3$.

Since deinterleaver is inverse of interleaver, the filters on the left are in series with the respective filters on the right

$h_0^+[n] = \delta[n]$ $\quad h_0^+[n+1] = \delta[n+1] \xleftrightarrow{DTFT} H_0^+(\omega) = e^{j\omega}$

From prob. 3 (d), we already have:

$h_1^+[n] * h_3^+[n] = h_3^+[n] * h_1^+[n] \xleftrightarrow{DTFT} e^{j\frac{3}{4}\omega} e^{\frac{1}{4}\omega} = e^{j\omega}$

In addition:

$h_2^+[n] * h_2^+[n] \xleftrightarrow{DTFT} e^{j\frac{\omega}{2}} e^{j\frac{\omega}{2}} = e^{j\omega}$

Since $BA = 4I$, we have:

$$z_k[n] = 4x_k[n+1], \quad k = 0, 1, 2, 3$$