NAME: SOLUTION KEY **29 Oct. 2012**

ECE 538 Digital Signal Processing I Exam 2 Fall 2012

Cover Sheet

WRITE YOUR NAME ON THIS COVER SHEET

Test Duration: 60 minutes.

Open Book but Closed Notes.

Calculators NOT allowed.

All work should be done in the space provided.

Problem 1. Consider a CT signal $x_a(t)$ with bandwidth (maximum frequency) W in Hz. The sampling rate is chosen to be above the Nyquist rate at $F_s = \frac{1}{T_s} = 3W$. $x_a(t)$ is reconstructed according to the formula below, where $W_1 < W_2$. Determine the respective values of W_1 and W_2, both in terms of W, so that the formula below yields perfect reconstruction.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h(t-nT_s) \quad \text{where: } h(t) = T_s \frac{1}{2W_1} \frac{\sin(2\pi W_1 t)}{\pi t} \frac{\sin(2\pi W_2 t)}{\pi t} \text{ and } F_s = \frac{1}{T_s} = 3W$$

The interpolating LPF $h(t)$ needs to have a flat spectrum ($=T_s$) up until the bandwidth of the signal, W:

$$W_2 - W_1 = W \quad \text{A}$$

Then it can roll-off linearly to 0 at $F_s - W$, the lower edge of the spectral replica centered at F_s:

$$W_2 + W_1 = F_s - W = 3W - W = 2W$$

$$\Rightarrow W_2 + W_1 = 2W \quad \text{B}$$

Two eqns in two unknowns.

$$\text{A} + \text{B} \Rightarrow 2W_2 = 3W \Rightarrow \boxed{W_2 = \frac{3}{2}W}$$

$$\text{A} \quad W_1 = W_2 - W = \frac{3}{2}W - \frac{2}{2}W \Rightarrow \boxed{W_1 = \frac{1}{2}W}$$

Problem 2. In class, we derived that the DTFT of $y[n] = x[Ln]$ is $Y(\omega) = \frac{1}{L}\sum_{k=0}^{L-1} X\left(\frac{\omega - k2\pi}{L}\right)$. Use this result in conjunction with the time-shift property of the DTFT to derive a similar general expression for the DTFT of $h_\ell^-[n] = h[Ln - \ell]$, denoted $H_\ell^-(\omega)$ in terms of the DTFT of $h[n]$, denoted $H(\omega)$. Show your work directly below and put a box around your final answer.

Define: $g_\ell^-[n] = h[n-\ell]$

Then: $h_\ell^-[n] = g_\ell^-[Ln]$

Time-Shift Property: $G_\ell^-(\omega) = H(\omega)e^{-j\ell\omega}$

Thus: $H_\ell^-(\omega) = \frac{1}{L}\sum_{k=0}^{L-1} G_\ell^-\left(\frac{\omega - k2\pi}{L}\right)$

Substituting:

$$H_\ell^-(\omega) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\ell\frac{(\omega - k2\pi)}{L}} H\left(\frac{\omega - k2\pi}{L}\right)$$

$$= \left\{\frac{1}{L}\sum_{k=0}^{L-1} e^{+j\frac{k\ell 2\pi}{L}} H\left(\frac{\omega - k2\pi}{L}\right)\right\} e^{-j\frac{\ell}{L}\omega}$$

$$= e^{-j\frac{\ell}{L}\omega}\left\{\frac{1}{L}\sum_{k=0}^{L-1} e^{j\frac{2\pi}{L}k\ell} H\left(\frac{\omega - k2\pi}{L}\right)\right.$$

Problem 3. Each of the four signals in this problem has the same bandwidth (that is, same maximum frequency) and is sampled at the same rate, the Nyquist rate (no aliasing.)

(a) Consider the continuous-time signal $x_0(t)$ below. A discrete-time signal is created by sampling $x_0(t)$ according to $x_0[n] = x_0(nT_s)$ for $T_s = \frac{2\pi}{20}$. Plot the magnitude of the DTFT of $x_0[n]$, $|X_0(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_0(t) = T_s \frac{1}{2}\left\{ \frac{\sin(10(t-\frac{\pi}{20}))}{\pi(t-\frac{\pi}{20})} + \frac{\sin(10(t+\frac{\pi}{20}))}{\pi(t+\frac{\pi}{20})} \right\}$$

$x_0[n] = x_0(nT_s) = x_0\left(n\frac{\pi}{10}\right)$ since $T_s = \frac{2\pi}{20} = \frac{\pi}{10}$

$$x_0[n] = \frac{\cancel{\pi}}{10}\cdot\frac{1}{2}\left\{ \frac{\sin\left(10\left(n\frac{\pi}{10}-\frac{\pi}{20}\right)\right)}{\cancel{\pi}\left(n\frac{\pi}{10}-\frac{\pi}{20}\right)} + \frac{\sin\left(10\left(n\frac{\pi}{10}+\frac{\pi}{20}\right)\right)}{\cancel{\pi}\left(n\frac{\pi}{10}-\frac{\pi}{20}\right)} \right\}$$

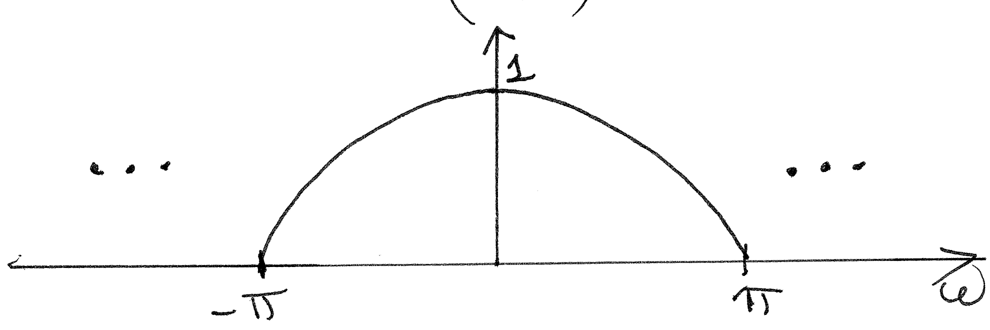
$$= \frac{1}{2}\left\{ \frac{\sin\left(\pi\left(n-\frac{1}{2}\right)\right)}{\pi\left(n-\frac{1}{2}\right)} + \frac{\sin\left(\pi\left(n+\frac{1}{2}\right)\right)}{\pi\left(n+\frac{1}{2}\right)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{\sin\left(\frac{\pi}{2}(2n-1)\right)}{\frac{\pi}{2}(2n-1)} + \frac{\sin\left(\frac{\pi}{2}(2n+1)\right)}{\frac{\pi}{2}(2n+1)} \right\}$$

for $|\omega| < \pi$:

$$X_0(\omega) = \frac{1}{2}\left\{ e^{-j\frac{1}{2}\omega} + e^{j\frac{1}{2}\omega} \right\}$$

$$= \cos\left(\frac{\omega}{2}\right)$$



(b) Consider the continuous-time signal $x_1(t)$ below. A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ for $T_s = \frac{2\pi}{20}$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

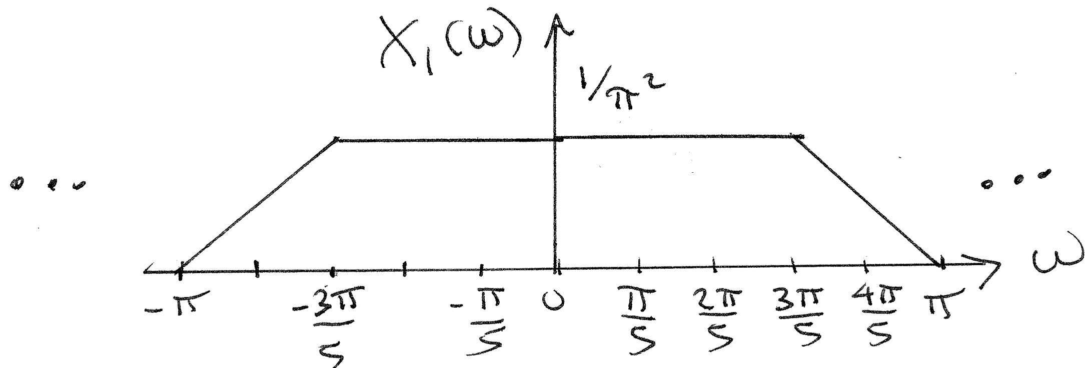
$$x_1(t) = T_s \frac{1}{2\pi} \left\{ \frac{\sin(2t)}{\pi t} \frac{\sin(8t)}{\pi t} \right\}$$

$$x_1[n] = x_1\left(n\frac{\pi}{10}\right) = \frac{\pi}{10} \frac{1}{2\pi} \left\{ \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n \frac{\pi}{10}} \frac{\sin\left(\frac{4\pi}{5}n\right)}{\pi n \frac{\pi}{10}} \right\}$$

$$= \frac{1}{20} \cdot 10^2 \cdot \frac{1}{\pi^2} \left\{ \frac{\sin\left(\frac{\pi}{5}n\right)}{n\pi} \frac{\sin\left(\frac{4\pi}{5}n\right)}{n\pi} \right\}$$

$$= \frac{5}{\pi^2} \underbrace{\left\{ \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n} \frac{\sin\left(\frac{4\pi}{5}n\right)}{\pi n} \right\}}_{\text{in freq. domain, height of trapezoid is } \frac{\pi/5}{\pi} = \frac{1}{5}}$$

$\Rightarrow$ so the height is $1/\pi^2$ (my bad :))



For rest of exam, I will assume that

$$x_1(t) = T_s \frac{\pi}{2} \left\{ \frac{\sin(2t)}{\pi t} \frac{\sin(8t)}{\pi t} \right\}$$

so that the height is 1 (unity)

(c) Consider the continuous-time signal $x_2(t)$ below. A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2(nT_s)$ for $T_s = \frac{2\pi}{20}$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

$$x_2(t) = T_s \frac{\pi}{5} \left\{ \frac{\sin(5t)}{\pi t} \right\}^2$$

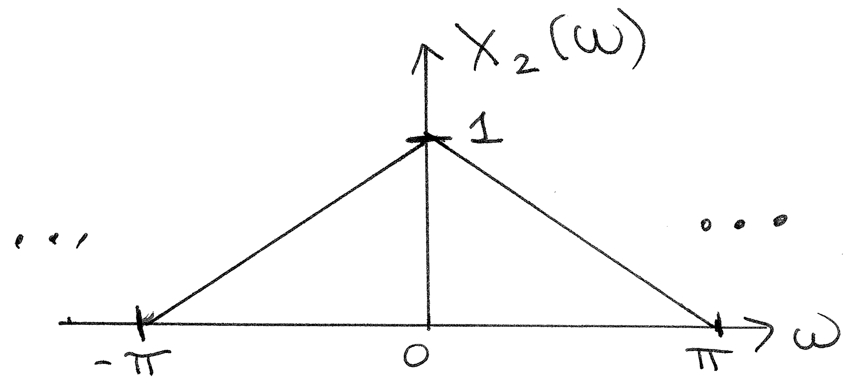
$$x_2[n] = \frac{\pi}{10}\frac{\pi}{5}\left(\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi \frac{\pi n}{10}}\right)^2$$

$$= \underbrace{\frac{\pi^2}{50}\frac{10^2}{\pi^2}}_{=2}\underbrace{\left\{\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\right\}^2}$$

in freq. domain, height of triangle is

$$\frac{\pi}{2}\Big/\pi = \frac{1}{2}$$

$\Rightarrow$ So, the height of the triangle is 1



(d) Consider the continuous-time signal $x_3(t)$ below. A discrete-time signal is created by sampling $x_3(t)$ according to $x_3[n] = x_3(nT_s)$ for $T_s = \frac{2\pi}{20}$. Plot the magnitude of the DTFT of $x_3[n]$, $|X_3(\omega)|$, over $-\pi < \omega < \pi$. Show all work.

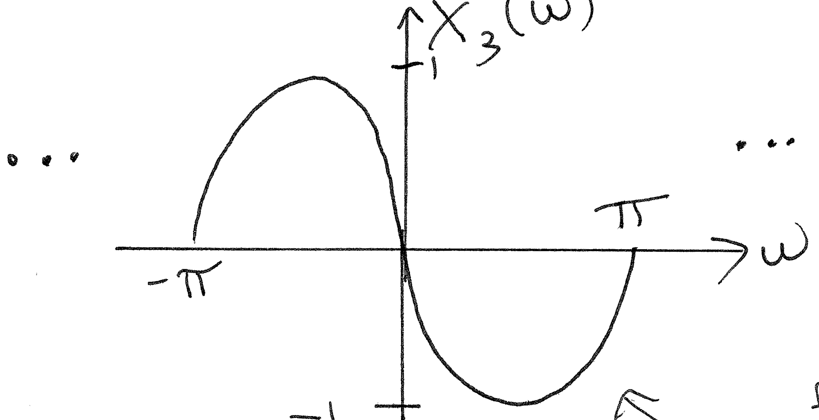
$$x_3(t) = T_s \frac{1}{2j}\left\{\frac{\sin(10(t-\frac{\pi}{10}))}{\pi(t-\frac{\pi}{10})} - \frac{\sin(10(t+\frac{\pi}{10}))}{\pi(t+\frac{\pi}{10})}\right\}$$

$$x_3[n] = \frac{\pi}{10}\frac{1}{2j}\left\{\frac{\sin\left(10\left(n\frac{\pi}{10}-\frac{\pi}{10}\right)\right)}{\pi\left(n\frac{\pi}{10}-\frac{\pi}{10}\right)} - \frac{\sin\left(10\left(n\frac{\pi}{10}+\frac{\pi}{10}\right)\right)}{\pi\left(n\frac{\pi}{10}+\frac{\pi}{10}\right)}\right\}$$

$$= \frac{\pi}{10}\cdot\frac{10}{\pi}\frac{1}{2j}\left\{\frac{\sin(\pi(n-1))}{\pi(n-1)} - \frac{\sin(\pi(n+1))}{\pi(n+1)}\right\}$$

$$X_3(\omega) = \text{DTFT}\underbrace{\left\{\frac{\sin(\pi n)}{\pi n}\right\}}_{\delta[n]}\frac{1}{2j}\left\{e^{-j\omega} - e^{j\omega}\right\}$$

$$= -\sin(\omega)$$



For $|X_3(\omega)|$
"flip this up to be positive"

Problem 4 part (a). Show all work in the space below. For all parts of this problem, $h_{LP}[n] = 4\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$.

(a) (i) Determine and write a simplified expression for the DTFT, $H_2^+(\omega)$, of $h_2^+[n] = h_{LP}[4n+2]$ that holds for $-\pi < \omega < \pi$. Simplify as much as possible.

(ii) Plot the magnitude of $H_2^+(\omega)$ over $-\pi < \omega < \pi$.

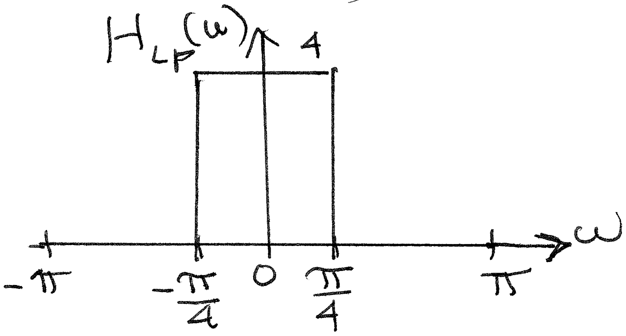
(iii) Plot the phase of $H_2^+(\omega)$ over $-\pi < \omega < \pi$.

From Problem 2 :

$$H_2^+(\omega) = \frac{1}{4}\sum_{k=0}^{3} \left\{ e^{-j\frac{2\pi}{4}(2)k} H_{LP}\left(\frac{\omega - k2\pi}{4}\right)\right\} e^{+j\frac{2}{4}\omega}$$

Since :

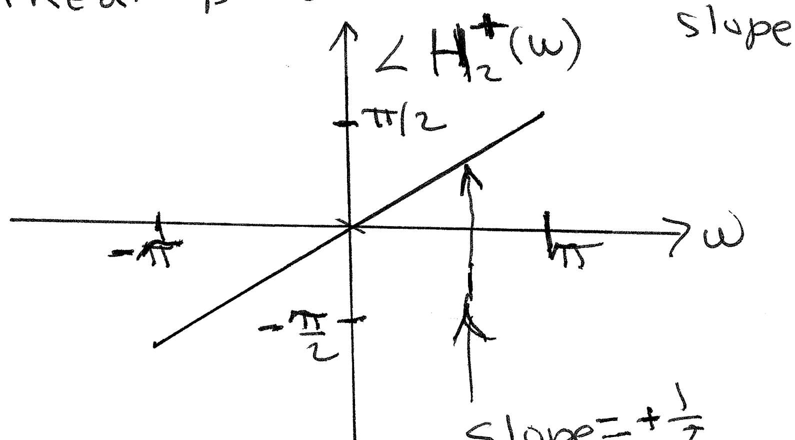
$$4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \overset{DTFT}{\longleftrightarrow}$$



Thus, only $k=0$ term is sum above contributes over $-\pi < \omega < \pi$.

Thus, $H_2^+(\omega) = e^{+j\frac{1}{2}\omega}$ over $|\omega| < \pi$

all-pass magnitude and linear phase with fractional slope



(b) Express the output of the filter $h_2^+[n] = h_{LP}[4n+2]$ in terms of sampled and time-shifted versions of the original analog input signals $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$.

You don't need to write out the expressions for $x_0(t)$, $x_1(t)$, $x_2(t)$, and $x_3(t)$. Also, just carry T_s along as a variable, rather than having to write its specific numerical value of $T_s = \frac{2\pi}{20}$. You don't have to do a lot of work here; just briefly explain your answer.

The filter $h_2^+[n]$ induces a fractional delay of $\frac{T_s}{2}$ back in the analog domain (effectively)

Since the input to $h_2^+[n]$ is

$y_2[n] = x_0[n] - x_1[n] + x_2[n] - x_3[n]$

where $x_i[n] = x_i(nT_s)$, then the output of the filter $h_2^+[n]$ is

$$x_0\left(\frac{T_s}{2} + nT_s\right) - x_1\left(\frac{T_s}{2} + nT_s\right) + x_2\left(\frac{T_s}{2} + nT_s\right) - x_3\left(\frac{T_s}{2} + nT_s\right)$$

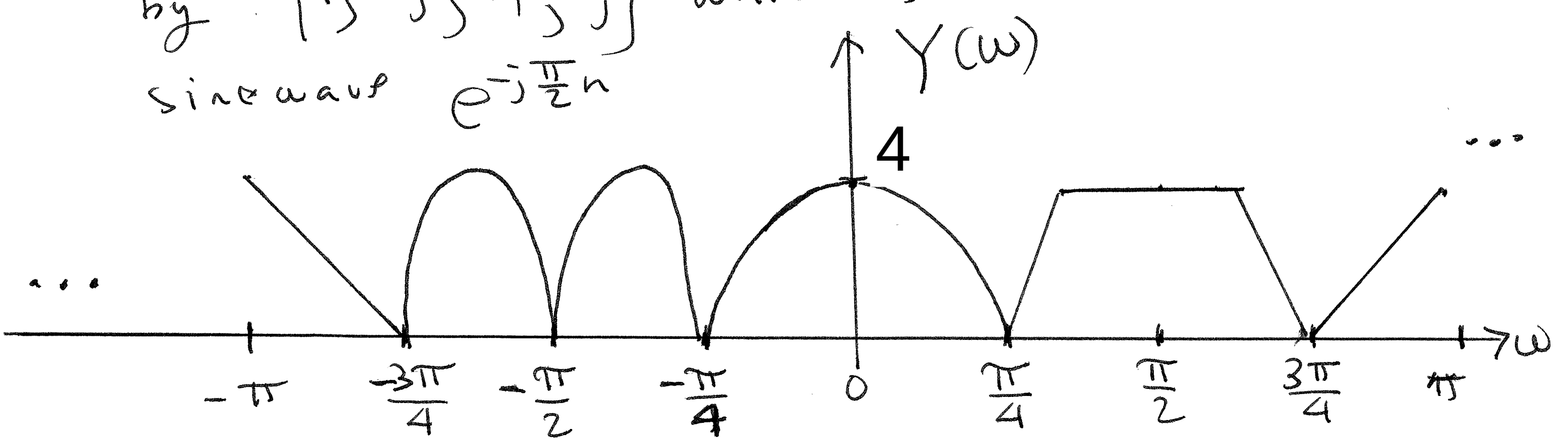
(c) Plot the magnitude of the DTFT, $Y(\omega)$, of the interleaved signal $y[n]$. Carefully label and graph the plot, clearly demarcating the subbands and showing what's in each subband.

- Signal $x_0(t)$ is effectively sampled at 4x Nyquist rate and occupies $|\omega| < \frac{\pi}{4}$

- Similarly, $x_1(t)$ occupies $\frac{\pi}{4} < \omega < \frac{3\pi}{4}$ in the frequency domain since its upsampled version is multiplied by $\{1, j, -1, -j\}$ every 4 samples $\Rightarrow$ $\left\{ \begin{array}{l} \text{one period of} \\ \text{sinewave } e^{j\frac{\pi}{2}n} \end{array} \right\}$

- $x_2(t)$ occupies $\frac{3\pi}{4} < \omega < \frac{5\pi}{5}$, which means half of it lies in $\frac{3\pi}{4} < \omega < \pi$, while the other half lies in $-\pi < \omega < -\frac{3\pi}{4}$

- $x_3(t)$ occupies $-\frac{3\pi}{4} < \omega < \frac{-\pi}{4}$ (center $= \frac{-\pi}{2}$) since its polyphase components are multiplied by $\{1, -j, -1, j\}$ which is one period of the sinewave $e^{-j\frac{\pi}{2}n}$



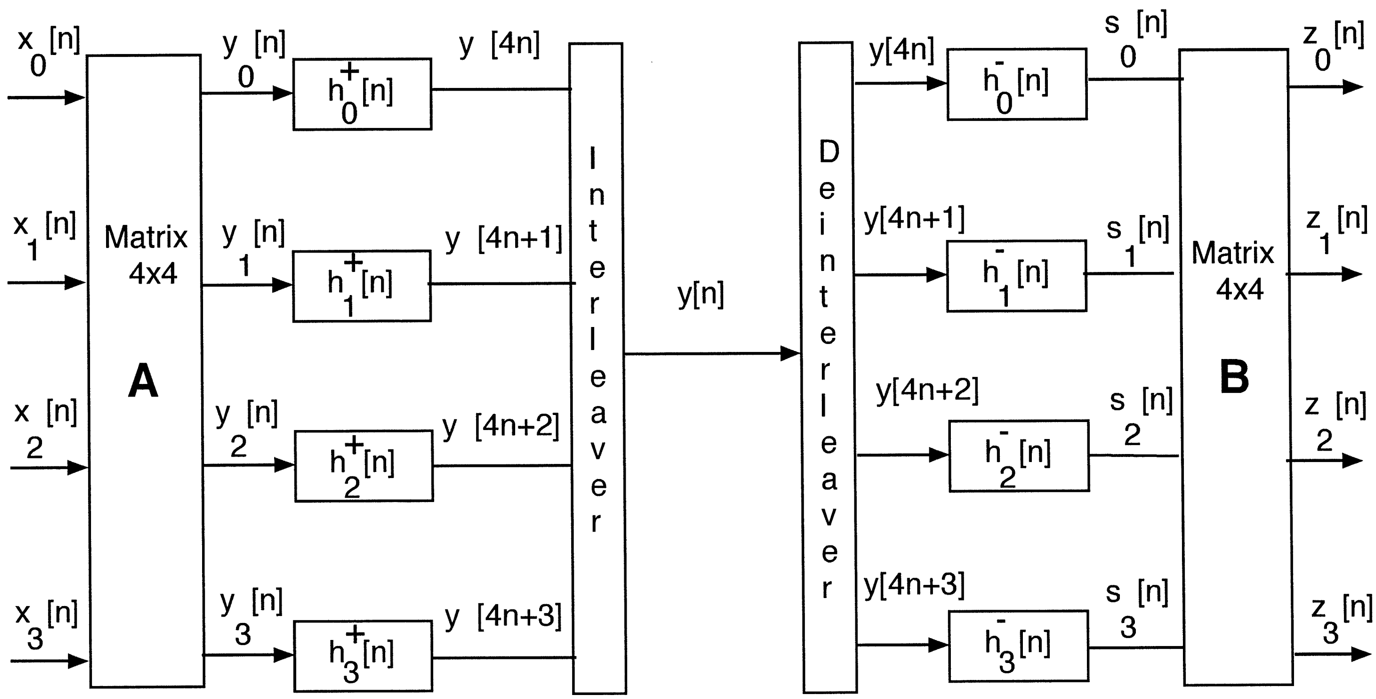


Figure 1.

Problem 4. This problem is about digital subbanding of the four DT signals $x_i[n]$, $i = 1, 2, 3, 4$ from Problem 3. Digital subbanding of these four signals is effected in the efficient way via filter bank in Figure 1. All of the quantities in Figure 1 are defined below.

Relative to Figure 1, the respective impulse responses of the polyphase component filters are defined in terms of the lowpass filter with the ideal impulse response below.

$$h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \tag{1}$$

The polyphase component filters on the left side of Figure 1 are defined as

$$h_\ell^+[n] = h_{LP}[4n + \ell], \qquad \ell = 0, 1, 2, 3. \tag{2}$$

The respective signals at the inputs to these filters are formed from the input signals (from Problem 3) via the matrix transformation below

$$\begin{bmatrix} y_0[n] \\ y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} x_0[n] \\ x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \tag{3}$$

The polyphase component filters on the right side of Figure 1 are defined as

$$h_\ell^-[n] = h_{LP}[4n - \ell], \qquad \ell = 0, 1, 2, 3. \tag{4}$$

The final output signals (on the far right side of Figure 1) are formed from linear combinations of the outputs of these filters via the matrix transformation below.

$$\begin{bmatrix} z_0[n] \\ z_1[n] \\ z_2[n] \\ z_3[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{5}$$

(d) Determine the convolution of $h_2^+[n] = h_{LP}[4n+2]$ and $h_2^-[n] = h_{LP}[4n-2]$, where $h_{LP}[n] = 4\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}$.

$$g_\ell[n] = h_2^+[n] * h_2^-[n] = h_{LP}[4n+2] * h_{LP}[4n-2] = ?$$

This is most easily determined via frequency domain analysis. You must show and explain your work. Do a stem plot of $g_\ell[n] = h_2^+[n] * h_2^-[n]$.

For this $h_{LP}[n]$, over $|\omega| < \pi$, we have

$$H_2^+(\omega) = e^{+j\frac{2}{4}\omega} \qquad H_2^-(\omega) = e^{-j\frac{2}{4}\omega}$$

Since convolution-in time $\Leftrightarrow$ multiplication in frequency

$$G_\ell(\omega) = e^{j\frac{\omega}{2}} e^{-j\frac{\omega}{2}} = 1 \quad \forall \omega$$

Thus, $g_\ell[n] = \delta[n]$

(e) Express the output of the filter $z_2[n]$ in terms of $x_0[n]$, $x_1[n]$., $x_2[n]$, and $x_3[n]$.

Since interleaver - deinterleaver tormas an identity (inverse operations)

and since $h_\ell^+[n] * h_\ell^-[n] = \delta[n]$ and

$y_\ell[n] * \delta[n] = y_\ell[n]$

and since $\underline{B}\ \underline{A} = 4\ \underline{\underline{I}}$

$\uparrow$ 4x4 Identity matrix

$$\Rightarrow z_\ell[n] = 4 x_\ell[n], \quad \ell = 0, 1, 2, 3$$

$$z_2[n] = 4 x_2[n]$$