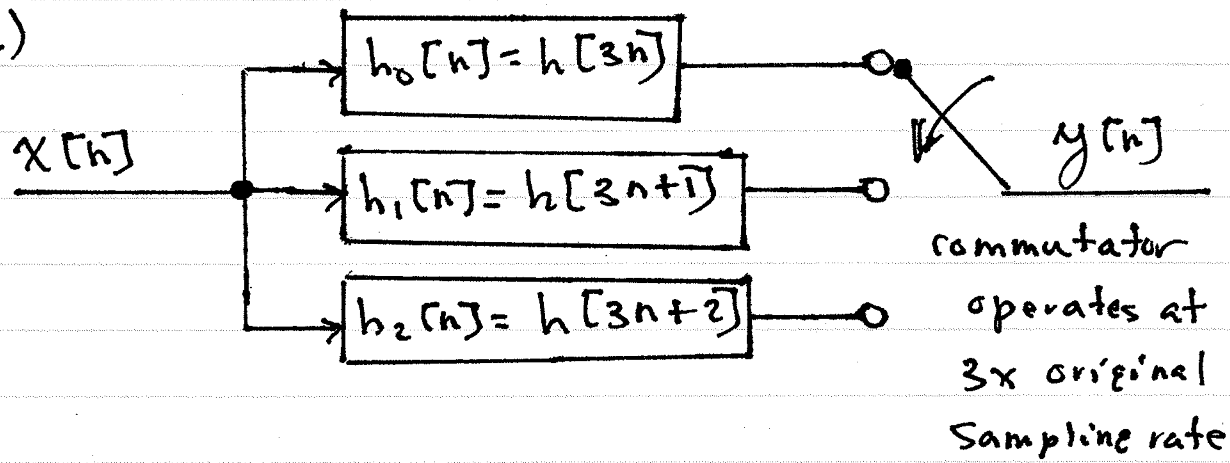
Solution to Exam 2 Fall 2009

Problem 1. Most of the solution to this problem comes from the set of notes labeled EfficientUpsamplingFinalWord.pdf, which is posted at the course web site.

(a)



For part (b), first recall

$$H_\ell(\omega) = \sum_{n=-\infty}^{\infty} h[Ln+\ell]\, e^{-j\omega n}$$

change of variables: $n' = Ln+\ell \Rightarrow n = \dfrac{n'-\ell}{L}$

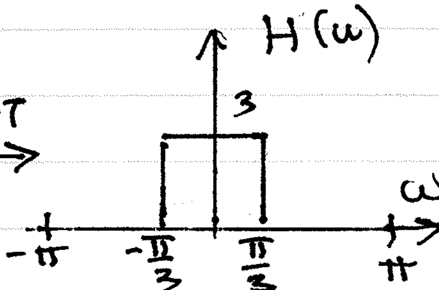
$$H_\ell(\omega) = \sum_{n'=nL+\ell} h[n']\, e^{-j\omega\frac{(n'-\ell)}{L}}$$

$$= \left\{ \sum_{n'=-\infty}^{\infty} \frac{1}{L} \sum_{k=0}^{L-1} e^{j2\pi\frac{k}{L}(n'-\ell)}\, h[n']\, e^{-j\frac{\omega}{L}n'} \right\} e^{j\frac{\omega\ell}{L}}$$

$$H_l(\omega) = \left\{ \frac{1}{L} \sum_{k=0}^{L-1} e^{-j2\pi \frac{l}{L} k} \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\frac{n(\omega - k2\pi)}{L}} \right\} e^{j\omega \frac{l}{L}}$$

$$h[Ln+l] \xleftrightarrow{\text{DTFT}} \left\{ \sum_{k=0}^{L-1} \frac{1}{L} e^{-j 2\pi \frac{l}{L} k} H\left(\frac{\omega - k2\pi}{L}\right) \right\} e^{j\omega \frac{l}{L}}$$

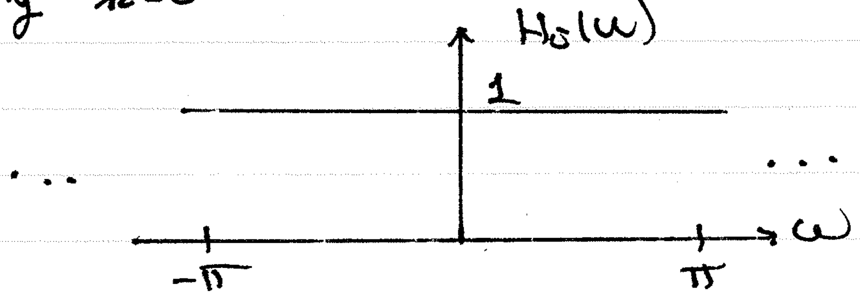
(b)-(i) $h[n] = 3 \dfrac{\sin\left(\frac{\pi}{3} n\right)}{\pi n} \xleftrightarrow{\text{DTFT}}$



$h_0[n] = h[3n] \Rightarrow l=0$ and $L=3$
in formula above

$$H_0(\omega) = \sum_{k=0}^{2} \frac{1}{3} H\left(\frac{\omega - k2\pi}{3}\right)$$

only $k=0$ term contributes over $-\pi < \omega < \pi$



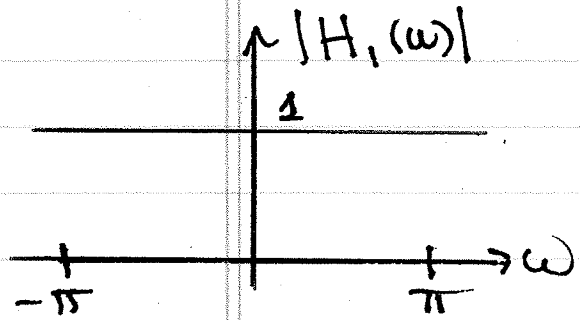
(b)-(ii) $l=1$, $L=3$

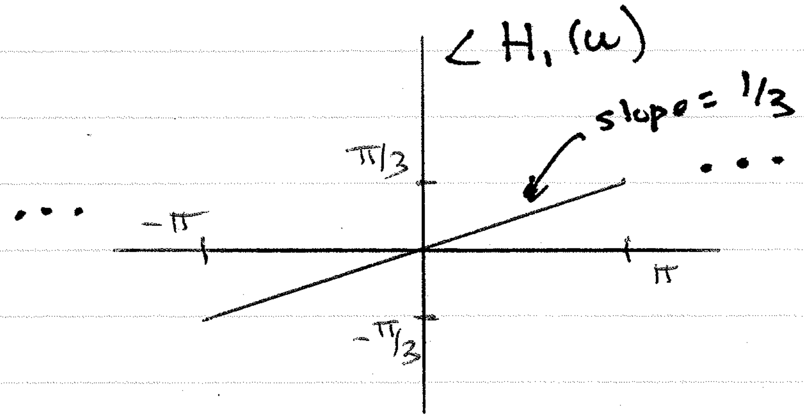
$$H_1(\omega) = \sum_{k=0}^{2} \frac{1}{3} e^{-j \frac{2\pi(1)}{3} k} H\left(\frac{\omega - k2\pi}{3}\right) e^{j\frac{\omega}{3}}$$

(b)-(iii) With $h[n] = 3\,\dfrac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$

only $k=0$ term contributes over $-\pi < \omega < \pi$

$$H_1(\omega) = e^{-j\frac{\omega}{3}} \quad \text{over } -\pi < \omega < \pi$$



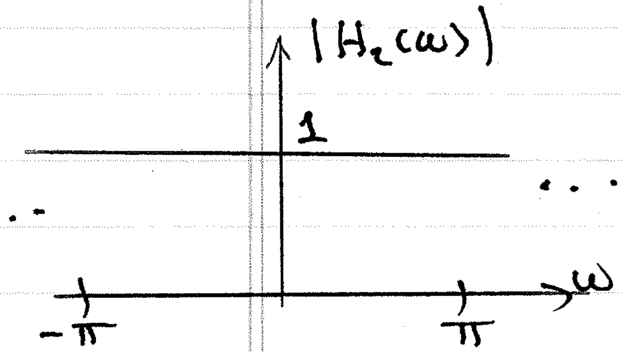


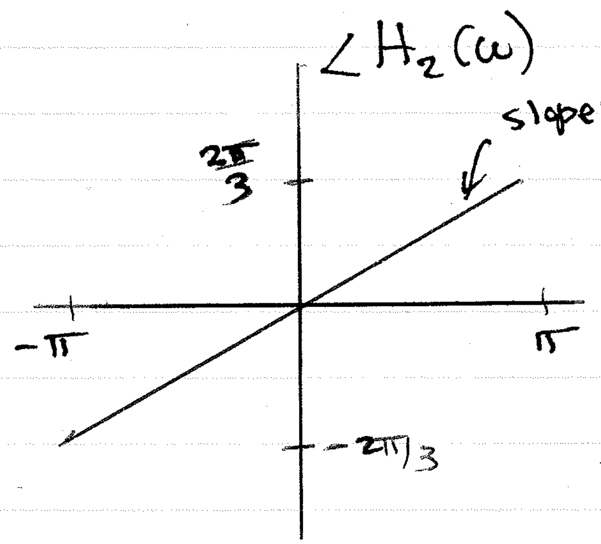
(iv) $l = 2$, $L = 3$

$$h[3n+2] \overset{DTFT}{\longleftrightarrow} \left\{ \frac{1}{3} \sum_{k=0}^{2} e^{-j2\pi\frac{2}{3}k}\, H\left(\frac{\omega - k2\pi}{3}\right) \right\} e^{j\omega\frac{2}{3}}$$

(v) For ideal case $h[n] = 3\,\dfrac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$ only $k=0$

term contributes over $-\pi < \omega < \pi \Rightarrow H_2(\omega) = e^{j\omega\frac{2}{3}}$





(c) $x[n] = x_a(n3)$

$$= \{\underset{\substack{\uparrow \\ n=0}}{0}, 3, 3, 3, 0\}$$

Assume aliasing effects are negligible

(c)-(i) The system in Fig. 1 increases the sampling rate by a factor of 3. Hence

$$y[n] = x_a\left(n\frac{3}{3}\right) = x_a(n)$$

$$= \{0, 1, 2, 3, 3, 3, 3, 3, 3, 3, 2, 1, 0\}$$

(c)-(ii) $y_0[n] = y[3n] = x[n] = \{\underset{\substack{\uparrow \\ n=0}}{0}, 3, 3, 3, 0\}$

(c)-(iii) $y_1[n] = y[3n+1] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 3, 3, 2, 0\}$

(c)-(iv) $y_2[n] = y[3n+2] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 3, 3, 1, 0\}$

Solution to Problem 2

$$x[n] \longrightarrow \boxed{h_i[n]} \longrightarrow \downarrow 3 \xrightarrow{w_i[n]} \uparrow 3 \xrightarrow{z_i[n]} \boxed{g_i[n]} \xrightarrow{y_i[n]}$$

$i = 0, 1, 2$

(a) $$W_i(\omega) = \frac{1}{3} \sum_{k=0}^{2} H_i\left(\frac{\omega - k2\pi}{3}\right) X\left(\frac{\omega - k2\pi}{3}\right)$$

$$Z_i(\omega) = W_i(3\omega)$$
$$= \frac{1}{3} \sum_{k=0}^{2} H_i\left(\omega - k\frac{2\pi}{3}\right) X\left(\omega - k\frac{2\pi}{3}\right)$$

$$Y(\omega) = \sum_{i=0}^{2} Y_i(\omega)$$
$$= \frac{1}{3} \sum_{i=0}^{2} \sum_{k=0}^{2} G_i(\omega) H_i\left(\omega - \frac{k2\pi}{3}\right) X\left(\omega - \frac{k2\pi}{3}\right)$$

$$= \frac{1}{3} \underbrace{\left\{ G_0(\omega) H_0(\omega) + G_1(\omega) H_1(\omega) + G_2(\omega) H_2(\omega) \right\}}_{A(\omega)} X(\omega)$$

$$+ \frac{1}{3} \underbrace{\left\{ G_0(\omega) H_0\left(\omega - \frac{2\pi}{3}\right) + G_1(\omega) H_1\left(\omega - \frac{2\pi}{3}\right) + G_2(\omega) H_2\left(\omega - \frac{2\pi}{3}\right) \right\}}_{B(\omega)} X\left(\omega - \frac{2\pi}{3}\right)$$

$$+ \frac{1}{3} \underbrace{\left\{ G_0(\omega) H_0\left(\omega - \frac{4\pi}{3}\right) + G_1(\omega) H_1\left(\omega - \frac{4\pi}{3}\right) + G_2(\omega) H_2\left(\omega - \frac{4\pi}{3}\right) \right\}}_{C(\omega)} X\left(\omega - \frac{4\pi}{3}\right)$$

Prob. (2)-(b) $\quad G_i(\omega) = H_i^*(\omega)$

$$A(\omega) = \frac{1}{3}\sum_{i=0}^{2} |H_i(\omega)|^2$$

$$B(\omega) = \frac{1}{3}\sum_{i=0}^{2} H_i\left(\omega - \frac{2\pi}{3}\right) H_i^*(\omega)$$

$$C(\omega) = \frac{1}{3}\sum_{i=0}^{2} H_i\left(\omega - \frac{4\pi}{3}\right) H_i^*(\omega)$$

(2)-(c) $\quad h_i[n] = e^{j(i)\frac{2\pi}{3}n} h(n) \quad , \quad i = 0, 1, 2$

$$H_i(\omega) = H\left(\omega + i\frac{2\pi}{3}\right)$$

$$A(\omega) = \frac{1}{3}\sum_{i=0}^{2} \left| H\left(\omega + i\frac{2\pi}{3}\right)\right|^2$$

minus

$$B(\omega) = \frac{1}{3}\sum_{i=0}^{2} H\left(\omega + (i-1)\frac{2\pi}{3}\right) H^*\left(\omega + i\frac{2\pi}{3}\right)$$

minus

$$C(\omega) = \frac{1}{3}\sum_{i=0}^{2} H\left(\omega + (i-2)\frac{2\pi}{3}\right) H^*\left(\omega + i\frac{2\pi}{3}\right)$$

$$H_i\left(\omega - \frac{4\pi}{3}\right) = H_i\left(\omega + 2\pi - \frac{4\pi}{3}\right) = H_i\left(\omega + \frac{6\pi}{3} - \frac{4\pi}{3}\right)$$

$$= H_i\left(\omega + \frac{2\pi}{3}\right)$$