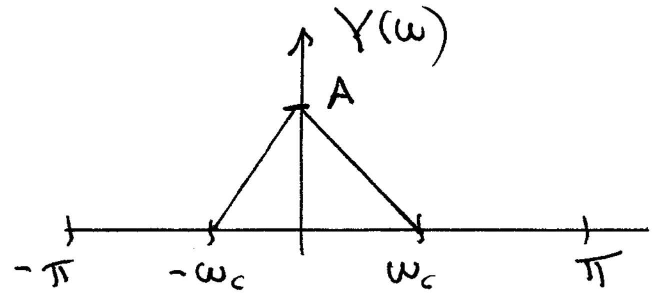
ECE 538 Solution to Exam 2 Fall 2007

Prob. 1

do (b) first: $x[n] = y[n] \cos(\omega_0 n)$

For illustrative purposes, use triangle spectrum for $y[n]$

$y[n] \overset{DTFT}{\longleftrightarrow}$

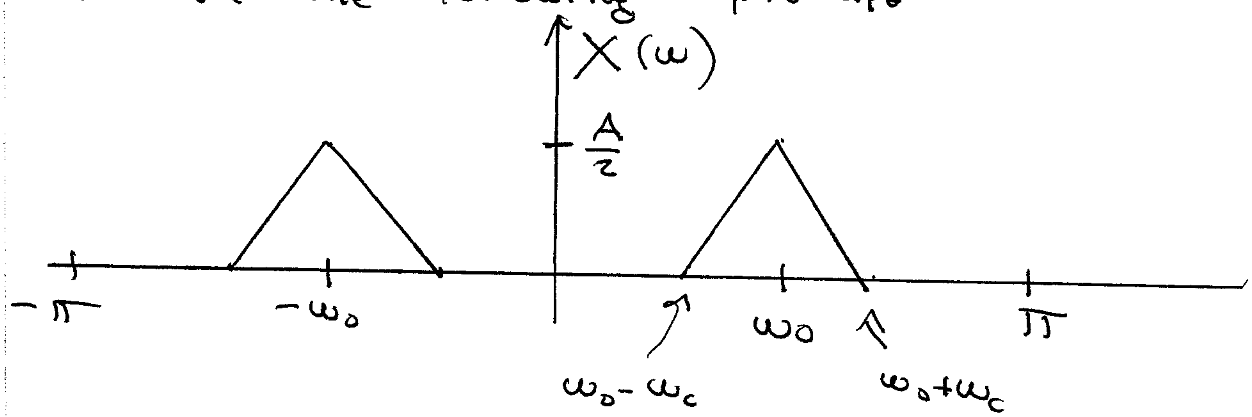


$$X(\omega) = \frac{1}{2} Y(\omega + \omega_0) + \frac{1}{2} Y(\omega - \omega_0)$$

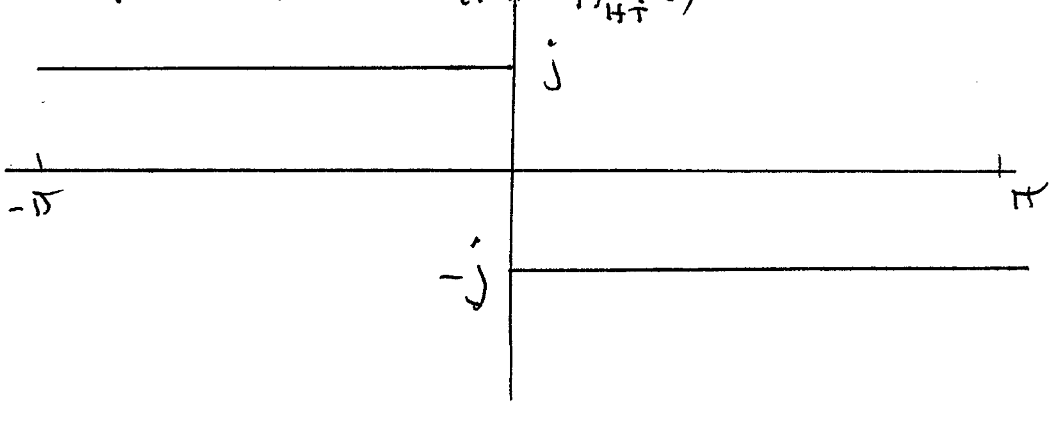
because of the conditions

$\omega_0 - \omega_c > 0$ and $\omega_0 + \omega_c < \pi$

we have the following "picture":



Now, multiply by frequency response of the Hilbert Transformer $H_{HT}(\omega)$



Sol'n to Prob. 1 (cont.)

The result is

$$X(\omega) = \frac{-j}{2} Y(\omega - \omega_0) + \frac{j}{2} Y(\omega + \omega_0)$$

This is exactly the same ~~thing~~ result obtained with

$$x[n] = y[n] \sin(\omega_0 n)$$

$$\xleftrightarrow{DTFT} \frac{1}{2j} Y(\omega - \omega_0) - \frac{1}{2j} Y(\omega + \omega_0)$$

Q.E.D

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(a) $$x[n] = \overbrace{2 \frac{\sin\left(\frac{\pi}{8} n\right)}{\pi n}}^{y[n]} \cos\left(\frac{3\pi}{8} n\right)$$

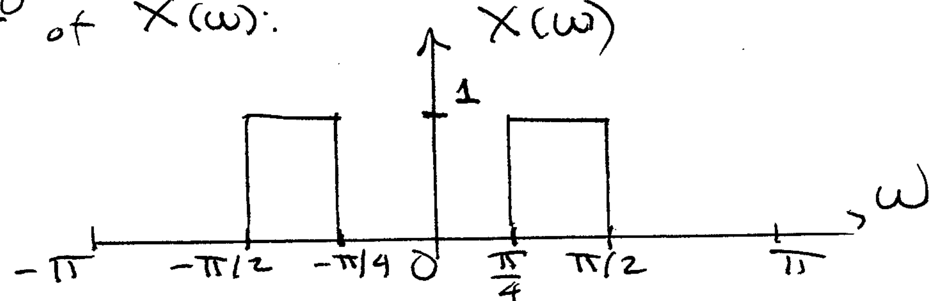
Since $\frac{3\pi}{8} - \frac{\pi}{8} = \frac{2\pi}{8} > 0$ $\quad (\omega_0 - \omega_c > 0)$

and $\frac{3\pi}{8} + \frac{\pi}{8} = \frac{4\pi}{8} < \pi$ $\quad (\omega_0 + \omega_c < \pi)$

Thus: $$\hat{x}[n] = 2 \frac{\sin\left(\frac{\pi}{8} n\right)}{\pi n} \sin\left(\frac{3\pi}{8} n\right)$$

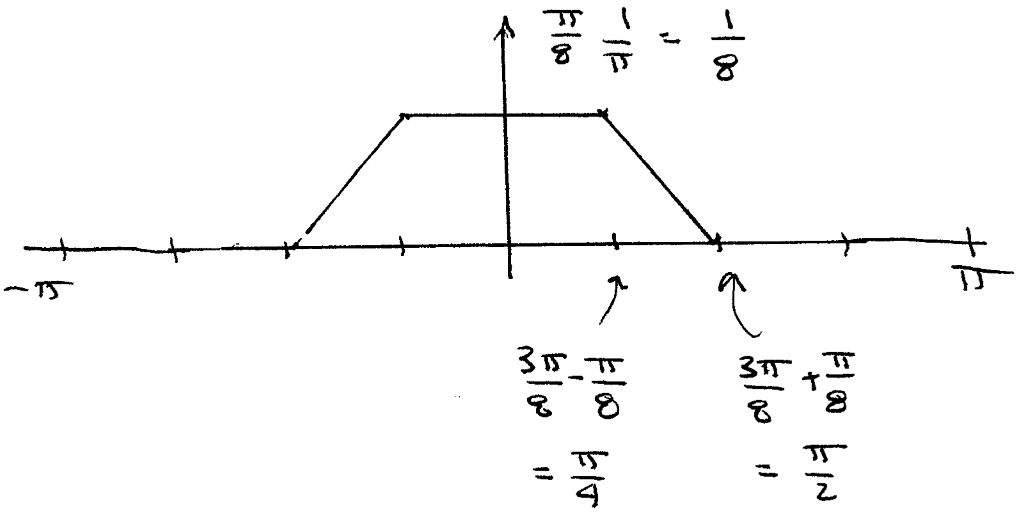
apply the results from part (b)

Plot of $X(\omega)$:

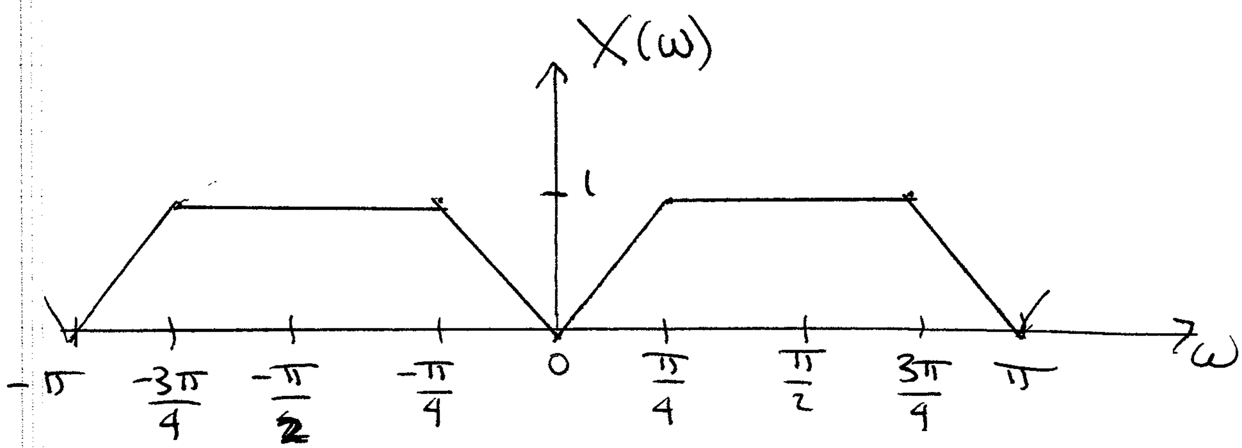


Soln to Prob. 1 (cont.)

(c) $$x[n] = 16 \left\{ \frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \, \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n} \right\} \cos\left(\frac{\pi}{2}n\right)$$



THUS:

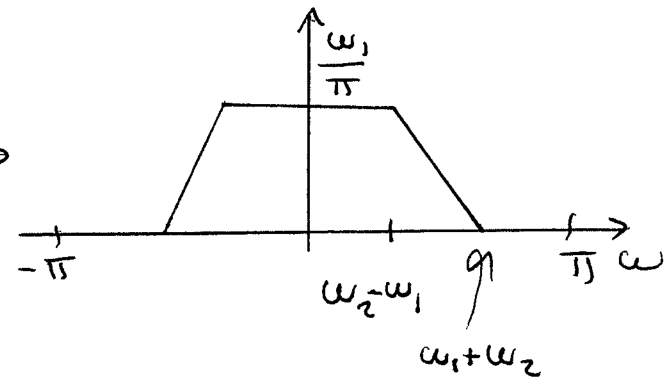


$$\hat{x}[n] = 16 \left\{ \frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \, \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n} \right\} \sin\left(\frac{\pi}{2}n\right)$$

Prob 2 Sol'n.

(a) Recall

$$\frac{\sin(\omega_1 n)}{\pi n}\,\frac{\sin(\omega_2 n)}{\pi n} \xleftrightarrow{\text{DTFT}}$$



$$0<\omega_1<\omega_2<\pi$$
$$\omega_1+\omega_2<\pi$$

Multiplying by $e^{j\frac{7\pi}{16}n}$ shifts spectrum to the right by $\frac{7\pi}{16}$ $\qquad e^{j\omega_0 n}x[n] \xleftrightarrow{\text{DTFT}} X(\omega-\omega_0)$

$$\omega_1=\frac{2\pi}{16} \qquad \omega_2=\frac{7\pi}{16}$$

$$\omega_2-\omega_1=\frac{5\pi}{16} \qquad \omega_1+\omega_2=\frac{9\pi}{16}$$

$$\frac{\omega_1}{\pi}=\frac{1}{\pi}\,\frac{2\pi}{16}=\frac{2}{16}$$

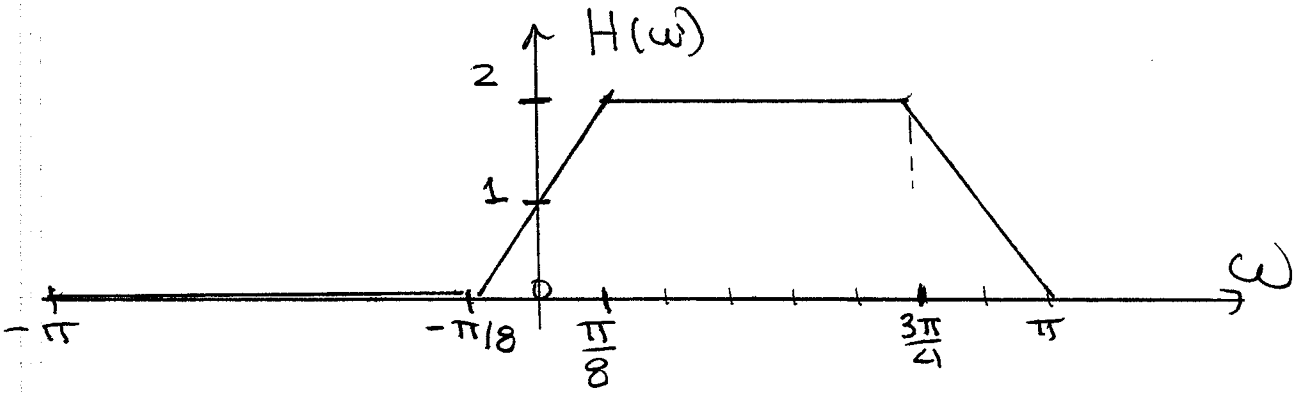
multiplying by 16 yields "height" of 2 in passband

now add $\frac{7\pi}{16}$:

$$\left\{\begin{array}{l} \omega_2-\omega_1+\frac{7\pi}{16}=\frac{12\pi}{16}=\frac{3\pi}{4} \\ \omega_1+\omega_2+\frac{7\pi}{16}=\frac{9\pi}{16}+\frac{7\pi}{16}=\frac{16\pi}{16}=\pi \end{array}\right.$$

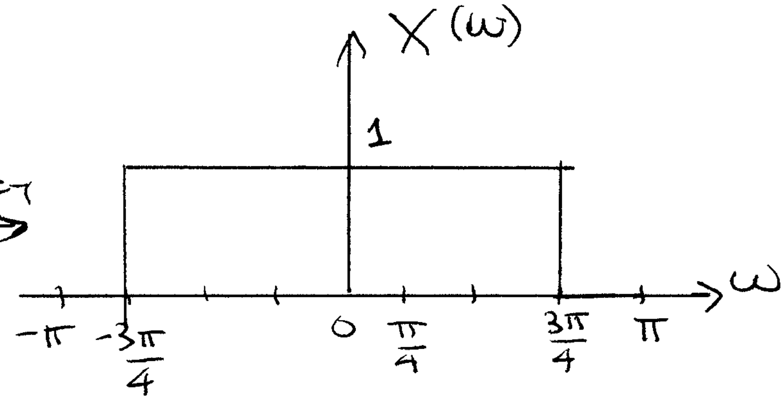
ALSO: $\frac{7\pi}{16}-(\omega_2-\omega_1)=\frac{7\pi}{16}-\frac{5\pi}{16}=\frac{2\pi}{16}=\frac{\pi}{8}$

$$\frac{7\pi}{16}-(\omega_1+\omega_2)=\frac{7\pi}{16}-\frac{9\pi}{16}=-\frac{\pi}{8}$$

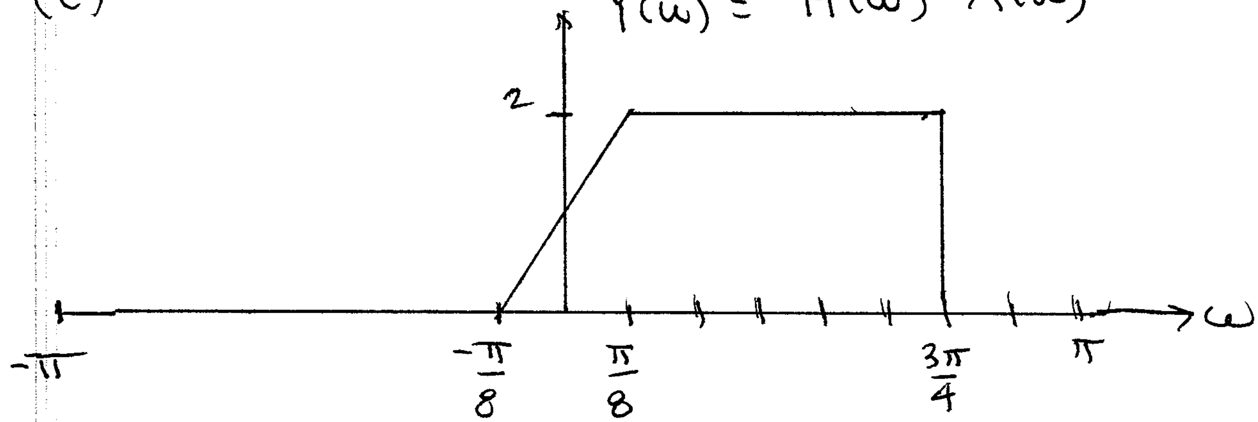


2 (b)

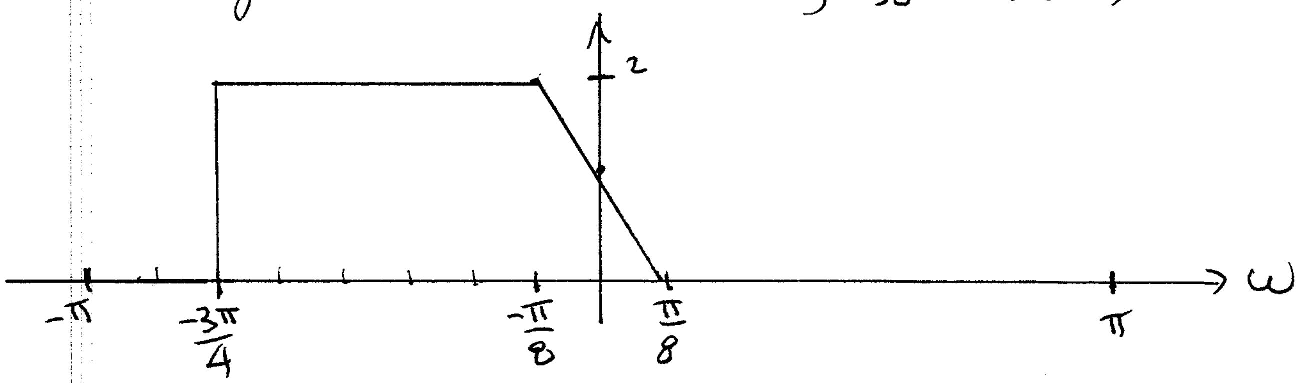
$$\frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n} \overset{DTFT}{\longleftrightarrow}$$



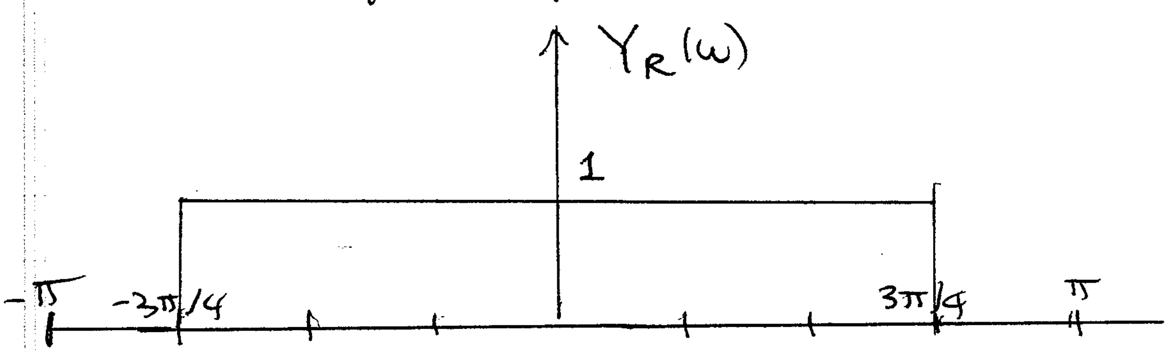
(c) $Y(\omega) = H(\omega)\,X(\omega)$



(d) $y^*[n] \overset{DTFT}{\longleftrightarrow} Y^*(-\omega)$ $\Big\}$ real-valued here so $Y(-\omega)$



(e) $y_R[n] = \frac{1}{2}\,y[n] + \frac{1}{2}\,y^*[n]$ $\Rightarrow$ summing answers to (c) and (d) and dividing by 2 gives back original spectrum for $x[n]$



2(f) $y_R[n] = x[n]$

2(g) $H(\omega) = 2 \qquad \Delta < \omega < \pi$

$\qquad\qquad = 0 \qquad -\pi < \omega < -\Delta$

Now, $Y(\omega) = H(\omega)\, X(\omega)$

$y^*[n] \xleftrightarrow{DTFT} Y^*(-\omega) = H^*(-\omega)\, X^*(-\omega)$

- for $x[n]$ real-valued $\Rightarrow$ $\underbrace{X^*(-\omega) = X(\omega)}$
- THUS:

$y_R[n] = \frac{1}{2} y[n] + \frac{1}{2} y^*[n] \xleftrightarrow{DTFT}$

$Y_R(\omega) = \frac{1}{2} H(\omega)\, X(\omega) + \frac{1}{2} H^*(-\omega)\, X(\omega)$

$= \frac{1}{2} \left\{ H(\omega) + H^*(-\omega) \right\} X(\omega)$

- require $\frac{1}{2} \left\{ H(\omega) + H^*(-\omega) \right\} = 1 \quad \forall\, \omega$
- THUS, the requirement for $-\Delta \leq \omega < \Delta$ is

$$\frac{1}{2} \left\{ H(\omega) + H^*(-\omega) \right\} = 1$$

- This can be satisfied in a variety of ways including the linear roll-off in this problem

Sol'n. to Prob. 3

$z_1 = j$, $z_2 = -j$ — zeroes of analog filter

$P_1 = -\frac{3}{5} + j\frac{4}{5}$, $P_2 = -\frac{3}{5} - j\frac{4}{5}$ — poles of analog filter

$$s = \frac{z-1}{z+1}$$

$$s(z+1) = (z-1)$$

$$z(s-1) = -s-1$$

$$\boxed{z = \frac{1+s}{1-s}}$$

(a) poles of $H_a(s)$ are in LHP (real part < 0)
$\Rightarrow$ stable analog filter
$\Rightarrow$ bilinear transform guarantees stable analog filter mapped to stable digital filter

(b) zeroes of $H_a(s)$ are on the imaginary axis in the s-plane
$\Rightarrow$ proved in class that the bilinear transform maps $s = j\Omega$ axis one-to-one to the unit circle $z = e^{j\omega}$ in the z-plane

- The mapping between analog frequency Ω and DT frequency is

$$\omega = 2\tan^{-1}(\Omega)$$

- a zero in s-plane at j, corresponds to a null at $\Omega = 1$ $\Rightarrow$ this is mapped to a null at $\omega = 2\tan^{-1}(1) = 2\frac{\pi}{4} = \frac{\pi}{2}$ with the digital filter

answer $\Rightarrow$ $\omega = \frac{\pi}{2}$

Prob. 3 Sol'n (cont.)

Parts (a) and (b) could have been alternatively solved by first using $z = \frac{1+s}{1-s}$ to find out where the analog poles and zeroes are mapped to in the z-plane

Poles: $$p_1^z = \frac{1 + \left(-\frac{3}{5} + j\frac{4}{5}\right)}{1 - \left(-\frac{3}{5} + j\frac{4}{5}\right)} = \frac{\frac{2}{5} + j\frac{4}{5}}{\frac{8}{5} - j\frac{4}{5}} \cdot \frac{5}{5}$$

$$= \frac{2 + j4}{8 - j4} = \frac{1 + 2j}{4 - 2j} = \frac{1}{2}\,\frac{1 + 2j}{2 - j}$$

$$= \frac{1}{2}\,\frac{1 + 2j}{2 - j}\,\frac{(2+j)}{(2+j)} = \frac{1}{2}\,\frac{(2-2) + j5}{5} = \frac{j}{2}$$

$$-\frac{3}{5} \pm j\frac{4}{5} \Rightarrow \text{mapped to: } \pm\frac{j}{2}$$

Since $\frac{j}{2}$ and $\frac{-j}{2}$ are inside unit circle

$\Rightarrow$ digital filter is stable

Zeroes: $$z_1^z = \frac{1+j}{1-j} \cdot \frac{(1+j)}{(1+j)} = \frac{(1-1) + j2}{2} = j$$

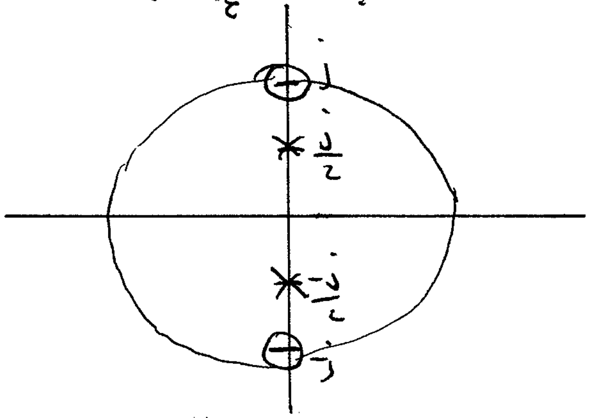
since $j = e^{j\frac{\pi}{2}}$ $\Rightarrow$ $\omega = \frac{\pi}{2}$ is notched out

that is, null at $\omega = \frac{\pi}{2}$

$H(\frac{\pi}{2}) = 0$

Prob. 3 (cont.)

(c) pole-zero diagram:

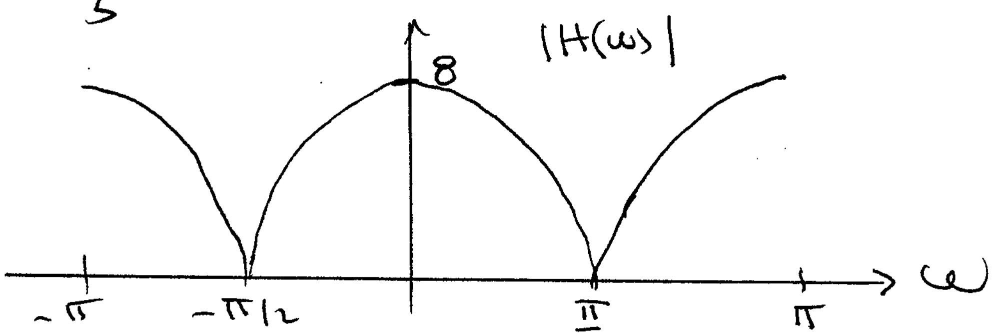


(d) $H(z) = G \dfrac{(z-j)(z+j)}{(z-\frac{j}{2})(z+\frac{j}{2})} = G \dfrac{(z^2+1)}{z^2+\frac{1}{4}}$

$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$

$H(0) = H(z)\Big|_{z=1} = G \dfrac{(1+1)}{1+\frac{1}{4}} = G \dfrac{2}{\frac{5}{4}} = \dfrac{8}{5} G$

$\dfrac{8}{5} G = 8 \Rightarrow G = 5$



$H(\pi) = H(z)\Big|_{z=-1} = 5 \dfrac{(1+1)}{1+\frac{1}{4}} = 8$

(d) difference eqn:

$$H(z) = 5\,\frac{(1 + z^{-2})}{1 + \frac{1}{4} z^{-2}} = \frac{Y(z)}{X(z)}$$

$$y[n] = -\frac{1}{4} y[n-2] + 5x[n] + 5x[n-2]$$