Cover Sheet

Write your name on this and every page
Test Duration: 60 minutes.
Coverage: Chapters 1-5.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains three problems.
Show your work in the space provided for each problem.
You must show all work for each problem to receive full credit.
Always simplify your answers as much as possible.

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Problem 1. [35 points]

(a) Consider two LTI systems connected in SERIES. System 1 has impulse response $h_1[n]$ below, where $p_1 = 0.5$ (in fractional form $p_1 = \frac{1}{2}$)

$$h_1[n] = \frac{1}{p_1} \left\{ \delta[n] + (p_1^2 - 1)p_1^n u[n] \right\}$$

System 2 has impulse response $h_2[n]$ below, where $p_2 = -0.5$

$$h_2[n] = \frac{1}{p_2} \left\{ \delta[n] + (p_2^2 - 1)p_2^n u[n] \right\}$$

(b) Determine the Z-Transform of the overall system. Draw the pole-zero diagram.

(c) Plot the magnitude of the frequency response $|H(\omega)|$ of the overall system over $-\pi < \omega < \pi$. Explain your answer.

(d) Write the difference equation for the overall system.

(e) Determine the output $y[n]$ of the overall system when the input is the sum of sinewaves (turned on forever) below

$$x[n] = 4 + 3(j)^n + 2(-j)^n + (-1)^n$$

(f) Extra Credit. Consider creating a new impulse response from your answer for part (a) as

$$g[n] = h[2n + 1] \quad \text{where: } \quad h[n] \quad \text{is the overall impulse response of the system}$$

That is, $g[n]$ is formed by keeping only the values of $h[n]$ for odd values of time, i.e., throwing the values of $h[n]$ for even values of time. Compute the energy of $g[n]$: 

$$E_g = \sum_{n=-\infty}^{\infty} g^2[n]$$
NAME:  

Page intentionally blank for Problem 1 Work

\( p_1 p_2 = -\frac{1}{4} \quad p_1 - p_2 = \frac{1}{2} - (-\frac{1}{2}) = 1 \quad p_2 - p_1 = -1 \)

(a) \( h[n] = h_1[n] * h_2[n] \)  

Series

\[
\frac{1}{p_1 p_2} \left\{ \delta[n] + \alpha p_1 u[n] + \alpha p_2 u[n] + \frac{\alpha_1 p_1}{p_1 - p_2} p_1^n u[n] + \frac{\alpha_2 p_2}{p_2 - p_1} p_2^n u[n] \right\} = \frac{1}{p_1 p_2} \left\{ \delta[n] + \left( \alpha + \frac{\alpha_1 \alpha_2 p_1}{p_1 - p_2} \right) p_1^n u[n] + \left( \alpha + \frac{\alpha_1 \alpha_2 p_2}{p_2 - p_1} \right) p_2^n u[n] \right\} = -4 \left\{ \delta[n] + \left( \alpha + \frac{9}{4} \right) \left( \frac{1}{2} \right)^n u[n] + \left( \alpha - \frac{9}{4} \right) \left(-\frac{1}{2} \right)^n u[n] \right\}
\]

\[
= -4 \left\{ \delta[n] + \left( \frac{15}{8} \right) \left[ \left( \frac{1}{2} \right)^n u[n] + \left(-\frac{1}{2} \right)^n u[n] \right] \right\}
\]

(b) \( H(z) = H_1(z) H_2(z) \)  

\( \alpha_1 = p_1^2 - 1 \quad \alpha_2 = p_2^2 - 1 \)

\[ H_1(z) = \frac{1}{p_1} \left\{ 1 + \alpha_1 \frac{z}{z - p_1} \right\} = \frac{1}{p_1} \left\{ \frac{z - p_1 + p_1^2 z - z}{z - p_1} \right\} = \frac{1}{p_1} \left\{ \frac{z - \frac{1}{p_1}}{z - p_1} \right\} = \frac{1}{2} \left\{ \frac{z - \frac{9}{4}}{z - \frac{1}{2}} \right\} \]

Similarly, replace \( p_1 \) with \( p_2 \):

\[ H_2(z) = \frac{1}{p_2} \left\{ \frac{z - \frac{1}{p_2}}{z - p_2} \right\} = \frac{1}{2} \left\{ \frac{z + \frac{1}{2}}{z - \frac{1}{2}} \right\} \]

Thus:\( H(z) = -4 \left\{ \frac{z - \frac{2}{z} - \frac{1}{z^2}}{z - \frac{1}{2}, \frac{z + \frac{1}{2}}{z + \frac{1}{2}}} \right\} = -\frac{1}{4} \left\{ \frac{z^2 - 4}{z^2 - \frac{1}{4}} \right\} \)
(c) \[ |H(\omega)| = \left| H_1(\omega) \right| \left| H_2(\omega) \right| \]

both are all-pass \( \Rightarrow \) product is all-pass (Problem on previous exam)

\[ \text{Plug in } z = 1 \text{ to find gain at } \omega = 0: \]
\[ H(z) \big|_{z=1} = -\frac{1}{4} \left( 1 - 4 \right) = -\frac{1}{4} \left( -\frac{3}{4} \right) = 1 \]

(d) \[ \frac{Y(z)}{X(z)} = -\frac{z^2 + 4}{z^2 - \frac{1}{4}} = -\frac{1 + 4z^{-2}}{1 - \frac{1}{4}z^{-2}} \]

\[ y[n] - \frac{1}{4} y[n-2] = -x[n] + 4x[n-2] \]

\[ y[n] = \frac{1}{4} y[n-2] = -x[n] + 4x[n-2] \]

(e) \text{sinewaves:} \]
\[ y[n] = H(0) x[n] + 3 H \left( \frac{\pi}{2} \right) (j)^n + 2 H \left( -\frac{\pi}{2} \right) (-j)^n \]

where \[ H(\omega) = DTFT \]
\[ = H(z) \big|_{z = e^{j\omega}} \]
(e) \[ H(\omega) \bigg|_{\omega=0} = H(z) \bigg|_{z=1} = 1 \text{ found previously} \]

\[ H(\omega) \bigg|_{\omega=\frac{\pi}{2}} = H(z) \bigg|_{z=j} = -4 \left\{ \frac{1 - 4}{1 - \frac{1}{4}} \right\} = -4 \left\{ \frac{5}{3} \right\} = -1 \]

Since poles are real-valued, \[ H(-\frac{\pi}{2}) = H(\frac{\pi}{2}) = -1 \]

\[ H(\omega) \bigg|_{\omega=\pi} = H(z) \bigg|_{z=-1} = -4 \left\{ \frac{1 - 4}{1 - \frac{1}{4}} \right\} = -4 \left\{ \frac{3}{4} \right\} = 1 \]

\[ y[n] = 4 + 3(-1) e^{j \frac{\pi}{2} n} + 2(-1) e^{-j \frac{\pi}{2} n} + e^{j \pi n} \]

(f) Extra Credit: for \( n > 0 \) \( (n \neq 0) \)
\[ h(n) = \frac{15}{8} \left( \left( \frac{1}{2} \right)^n + \left( -\frac{1}{2} \right)^n \right) u(n) \]

\( h(n) \) for \( n \) odd \( \Rightarrow 0 \)

\[ g(n) = h(2n+1) = 0 \text{ for all } n \]

\[ E_\varepsilon = 0 \]
(a) Consider \( h_1[n] \) and \( h_2[n] \) to be two distinct all-pass filters \((p_1 \neq p_2)\) with respective impulse responses below. Is the sum \( h[n] = h_1[n] + h_2[n] \) an all-pass filter for any and all values of \( p_1 \) and \( p_2 \) \((p_1 \neq p_2)\)? Explain your answer. Your explanation is much more important than your answer.

\[
h_1[n] = \frac{1}{p_1} \left\{ \delta[n] + (p_1^2 - 1)p_1^n u[n] \right\}
\]

\[
h_2[n] = \frac{1}{p_2} \left\{ \delta[n] + (p_2^2 - 1)p_2^n u[n] \right\}
\]

\[
h[n] = h_1[n] + h_2[n] \Rightarrow \text{parallel combination}
\]

Many past exams show that the parallel combination is either a notch-filter or digital resonator when \( p_2 = -p_1 \), \( \Rightarrow \) Not all-pass

Another argument:

\[
H(\omega) = |H_1(\omega)| e^{j \angle H_1(\omega)} + |H_2(\omega)| e^{j \angle H_2(\omega)}
\]

\[
= 1 \cdot \left( e^{j \angle H_1(\omega)} + e^{j \angle H_2(\omega)} \right)
\]

\[
\text{since } |H_1(\omega)| = 1 \quad \text{and} \quad |H_2(\omega)| = 2 \quad \forall \omega
\]

will have a magnitude that is Not flat, in general for all frequencies
Problem 2.(b)

(b) Consider \( h[n] \) to be an all-pass filter with respective impulse response below.

\[
h[n] = \frac{1}{p} \{ \delta[n] + (p^2 - 1)p^nu[n] \}
\]

\( p \) is real-valued \hspace{1cm} (3)

Is the product

\[
g[n] = e^{j\omega_0 n} h[n]
\]

an all-pass filter for any and all values of the frequency \( \omega_0 \)? Explain your answer. Your explanation is much more important than your answer.

From DTFT property:

\[
G(\omega) = H(\omega - \omega_0)
\]

\[
|G(\omega)| = |H(\omega - \omega_0)|
\]

since \( |H(\omega)| = 1 \) for all \( \omega \), \( -\infty < \omega < \infty \)

\[
|H(\omega - \omega_0)| = 1 \hspace{1cm} \text{for all } \omega \Rightarrow g[n] \text{ is all-pass}
\]

OR: can you result

\[
\mathcal{F}\{ h[n] \} = e^{j\omega_0 n} \mathcal{F}\{ h[n] \}
\]

since \( h[n] \) is all-pass

\[
= e^{j\omega_0 \ell} \delta(\ell)
\]

\[
= e^{j\omega_0 \ell} \delta(\ell)
\]

\[
= \delta(\ell) \Rightarrow \text{all-pass}
\]
Problem 3. Note: this problem is different from a similar problem on last year’s exam in that the two sequences, $x_1[n]$ and $x_2[n]$, are of different lengths.

(a) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-3 sequence $x_1[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_1[n] = \{1, 1, -1\} = \delta[n] + \delta[n-1] - \delta[n-2]$$

(b) Determine the autocorrelation $r_{x_2x_2}[\ell]$ of the length-5 sequence $x_2[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_2[n] = \{1, 1, -1, 1\} = \delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4]$$

(c) The sequence $x_1[n]$ defined above is input to the system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_1x_1}[\ell]$ between the output and input.

$$y_1[n] = 4x_1[n - 3] + x_1[n - 4]$$

(d) The sequence $x_2[n]$ defined above is input to the same system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_2x_2}[\ell]$ between the output and input.

$$y_2[n] = 4x_2[n - 3] + x_2[n - 4]$$

(e) Sum your answers to parts (c) and (d) to form the sum below. Do a stem plot of $r_{yx}[\ell]$.

$$r_{yx}[\ell] = r_{y_1x_1}[\ell] + r_{y_2x_2}[\ell]$$

```
(a) \[ \{1, 1, -1\} \ast \{\cdots, 1, 1\} \] \Rightarrow \text{Table Method}
and know max value
is at \ell = 0
```

\[
\begin{array}{cccc}
-1 & 0 & 3 & 0 & -1 \\
-1 & 1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
r_{x_1x_1}[\ell] = \{\cdots, -1, 0, 3, 0, -1\} \\
\end{array}
\]

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(b) \[ \{1, 1, 1, 1, -1, 1\} \times \{1, -1, 1, 1, 1\} \]

Table Method
max value is at \( l = 0 \)

\[ R_{x_2 x_2} [l] = \]

\[ \{0, 0, 0, 5, 0, 5\} \]

\( l = 0 \)

(c) \[ R_{y_1 y_1} [l] = 4R_{y_1 y_1} [l-3] + R_{y_1 y_1} [l-4] = h_{y_1 y_1} (l) \times R_{y_1 y_1} \]

\[ \{0, 0, 0, 4, 1\} \times \{-1, 0, 3, 0, 5\} \]

\( l = 0 \)

\[ \begin{array}{cccccc}
-4 & 0 & 12 & 0 & -4 & -1 \\
-1 & 0 & 3 & 0 & -1 & \\
0 & 0 & -4 & -1 & 12 & 3 \\
0 & 0 & 0 & -4 & -1 & -1 \\
\end{array} \]

\[ \{0, 0, 0, -4, -1, 12, 3, -4, -1\} \]
\((d) \quad r_{x_2 x_2}(l) = h_2(l) * r_{x_2 x_2}(l)
= 4r_{x_2 x_2}(l-3) + r_{x_2 x_2}(l-4)\)

\(\{0,0,0,4,1\} \leftarrow \{1,0,0,5,0,1,0,1\}\)

\[
\begin{array}{ccccccc}
4 & 0 & 4 & 0 & 2 & 0 & 6 \\
1 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 1 \\
\end{array}
\]

\(\{0,0,0,4,5,4,1,2,0,5,4,1,4,1\} = r_{x_2 x_2}(l)\)

\(\ell_{\text{ell}=0}\)

\[(e) \quad r_{x x}(l) = r_{x_1 x_1}(l) + r_{x_2 x_2}(l)\]

Due to linearity: define \(r_{\text{sum}}(l) = r_{x_1 x_1}(l) + r_{x_2 x_2}(l)\)

\[\{1,0,0,0,8,0,0,0,0,1\}\]

\(r_{x x}(l) = \{0,0,0,4,1\} \leftarrow \{1,0,0,0,8,0,0,0,0,1\}\)

\[
\begin{array}{ccccccc}
4 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 4 & 1 & 0 & 0 & 3 & 2 & 8 & 0 & 0 & 0 & 4 & 1 \\
\end{array}
\]

\(\ell_{\text{ell}=0}\)

\(\{0,0,0,4,1,0,0,3,2,8,0,0,0,4,1\} = \text{answer}\)