NAME: SOLUTION

with corrections Prob 1, Sept. 25, 2017

EE538 Digital Signal Processing I Fall 2015
Exam 1 Monday, Sept. 28, 2015

Cover Sheet

Write your name on this and every page
Test Duration: 60 minutes.
Coverage: Chapters 1-5.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **three** problems.
Show your work in the space provided for each problem.
You must show all work for each problem to receive full credit.
Always simplify your answers as much as possible.

Prob. No.	Topic(s)	Points
1.	Frequency Response and Interconnection of LTI Systems, Pole-Zero Diagrams	40
2.	LTI Systems: All-Pass Filters	20
3.	DT Autocorrelation, Cross-Correlation and their Related Properties	40

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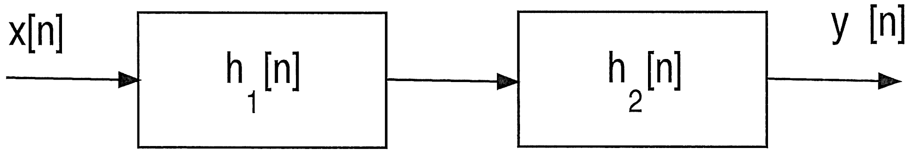
Problem 1. [35 points]

(a) Consider two LTI systems connected in SERIES. System 1 has impulse response $h_1[n]$ below, where $p_1 = 0.5$ (in fractional form $p_1 = \frac{1}{2}$)

$$\text{System 1:} \qquad h_1[n] = \frac{1}{p_1}\left\{\delta[n] \; + (p_1^2 - 1)p_1^n u[n]\right\}$$

System 2 has impulse response $h_2[n]$ below, where $p_2 = -0.5$

$$\text{System 2:} \qquad h_2[n] = \frac{1}{p_2}\left\{\delta[n] \; + (p_2^2 - 1)p_2^n u[n]\right\}$$



(a) Determine a closed-form expression for the impulse response, $h[n]$, of the overall system. Write how $h[n]$ is related to $h_1[n]$ and $h_2[n]$, and then show all work in determining your final answer. You may want to make use of the convolution result below. Note: you can solve the rest of this problem (except the extra credit) without the answer to part (a.)

$$\alpha^n u[n] * \beta^n u[n] = \frac{\alpha}{\alpha - \beta}\alpha^n u[n] + \frac{\beta}{\beta - \alpha}\beta^n u[n]$$

(b) Determine the Z-Transform of the overall system. Draw the pole-zero diagram.

(c) Plot the magnitude of the frequency response $|H(\omega)|$ of the overall system over $-\pi < \omega < \pi$. Explain your answer.

(d) Write the difference equation for the overall system.

(e) Determine the output $y[n]$ of the overall system when the input is the sum of sinewaves (turned on forever) below

$$x[n] = 4 + 3(j)^n + 2(-j)^n + (-1)^n$$

(f) **Extra Credit.** Consider creating a new impulse response from your answer for part (a) as

$$g[n] = h[2n+1] \qquad \text{where:} \quad h[n] \quad \text{is the overall impulse response of the system}$$

That is, $g[n]$ is formed by keeping only the values of $h[n]$ for odd values of time, i.e., throwing the values of $h[n]$ for even values of time. Compute the energy of $g[n]$:

$$E_g = \sum_{n=-\infty}^{\infty} g^2[n]$$

$p_1 p_2 = -\frac{1}{4}$ $\qquad p_1 - p_2 = \frac{1}{2} - (-\frac{1}{2}) = 1$ $\qquad p_2 - p_1 = -1$

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$\alpha_1 = p_1^2 - 1 = \frac{-3}{4}$ =α

$\alpha_2 = p_2^2 - 1 = \frac{-3}{4}$ =α

The value of α above has nothing to do with the α in the basic convolution formula result given in part (a)

(a) $h[n] = h_1[n] * h_2[n]$ Series

$$= \frac{1}{p_1 p_2} \{\delta[n] + \alpha_1 p_1^n u[n]\} * \{\delta[n] + \alpha_2 p_2^n u[n]\}$$

$$= \frac{1}{p_1 p_2} \left\{\delta[n] + \alpha p_1^n u[n] + \alpha p_2^n u[n] + \frac{\alpha_1 \alpha_2 p_1}{p_1 - p_2} p_1^n u[n] + \frac{\alpha_1 \alpha_2 p_2}{p_2 - p_1} p_2^n u[n]\right\}$$

$$= \frac{1}{p_1 p_2} \left\{\delta[n] + \left(\alpha + \frac{\alpha_1 \alpha_2 p_1}{p_1 - p_2}\right) p_1^n u[n] + \left(\alpha + \frac{\alpha_1 \alpha_2 p_2}{p_2 - p_1}\right) p_2^n u[n]\right\}$$

$$= -4 \left\{\delta[n] + \left(\alpha + \frac{9}{4}\frac{1}{2}\right)\left(\frac{1}{2}\right)^n u[n] + \left(\alpha + \frac{9}{4}\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n u[n]\right\}$$

$$= -4\, \delta[n] + \left(\frac{15}{8}\right)\left[\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{2}\right)^n u[n]\right]$$

(b) $H(z) = H_1(z) H_2(z)$ $\qquad \alpha_1 = p_1^2 - 1 \qquad \alpha_2 = p_2^2 - 1$

$$H_1(z) = \frac{1}{p_1}\left\{1 + \alpha_1 \frac{z}{z - p_1}\right\} = \frac{1}{p_1}\left\{\frac{z - p_1 + p_1^2 z - z}{z - p_1}\right\}$$

$$= \left\{\frac{-1 + p_1 z}{z - p_1}\right\} = p_1 \left\{\frac{z - \frac{1}{p_1}}{z - p_1}\right\} = \frac{1}{2}\left\{\frac{z - 2}{z - \frac{1}{2}}\right\}$$

Similarly, replace p_1 with p_2:

$$H_2(z) = p_2 \left\{\frac{z - \frac{1}{p_2}}{z - p_2}\right\} = \frac{-1}{2}\left\{\frac{z + 2}{z + \frac{1}{2}}\right\}$$

Thus: $H(z) = -\frac{1}{4}\left\{\frac{z - 2}{z - \frac{1}{23}}\right\}\left\{\frac{z + 2}{z + \frac{1}{2}}\right\} = -\frac{1}{4}\left\{\frac{z^2 - 4}{z^2 - \frac{1}{4}}\right\}$

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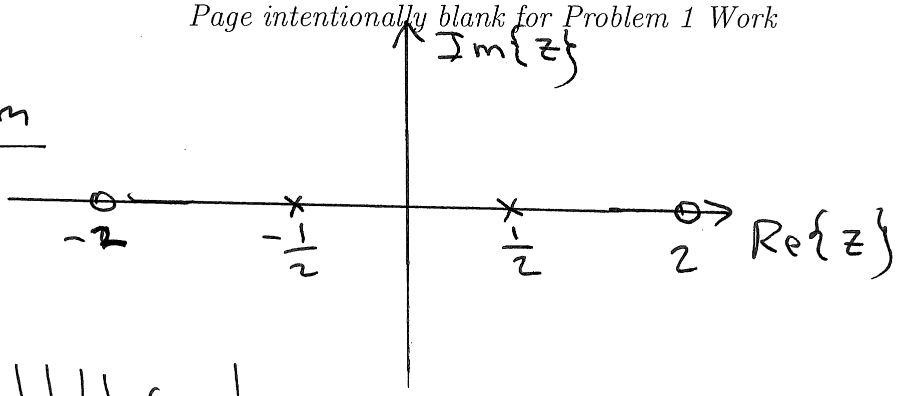
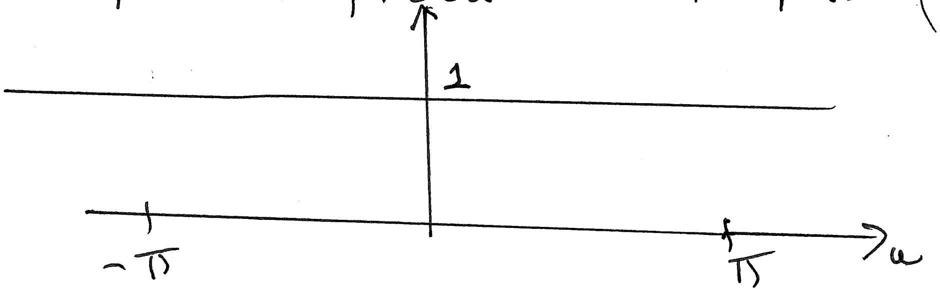
Pole-Zero Diagram



(c) $|H(\omega)| = |H_1(\omega)|\,|H_2(\omega)|$

both are all-pass ⟹ product is all-pass (problem on previous exam)



plug in $z=1$ to find gain at $\omega=0$:

$$H(z)\Big|_{z=1} = -\frac{1}{4}\left\{\frac{1-4}{1-\frac{1}{4}}\right\} = -\frac{1}{4}\left\{\frac{-3}{3/4}\right\} = 1$$

(d) $$\frac{Y(z)}{X(z)} = \frac{-z^2+4}{z^2-\frac{1}{4}} = \frac{-1+4z^{-2}}{1-\frac{1}{4}z^{-2}}\,\frac{1}{4}$$

$$y[n] - \frac{1}{4}y[n-2] = -\frac{1}{4}x[n] + x[n-2]$$

$$y[n] = \frac{1}{4}y[n-2] - \frac{1}{4}x[n] + x[n-2]$$

(e) sinewaves:

$$y[n] = H(0)4 + 3H\left(\frac{\pi}{2}\right)(j)^n + 2H\left(-\frac{\pi}{2}\right)(-j)^n + H(\pi)(-1)^n$$

where $H(\omega) = \text{DTFT}$
$= H(z)_{z=e^{j\omega}}$

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(e) $H(\omega)\Big|_{\omega=0} = H(z)\Big|_{z=1} = 1$ found previously

$$H(\omega)\Big|_{\omega=\pi/2} = H(z)\Big|_{z=j} = -4\left\{\frac{-1-4}{-1-\frac{1}{4}}\right\} = -\frac{1}{4}\left\{\frac{5}{\frac{5}{4}}\right\} = -1$$

Since poles are real-valued $H(-\frac{\pi}{2}) = H^*(\frac{\pi}{2}) = -1$

at $\omega=\pi$ $$H(\omega)\Big|_{\omega=\pi} = H(z)\Big|_{z=-1} = -\frac{1}{4}\left\{\frac{1-4}{1-\frac{1}{4}}\right\} = -\frac{1}{4}\left\{\frac{-3}{\frac{3}{4}}\right\} = 1$$

$$y[n] = 4 + 3(-1)e^{j\frac{\pi}{2}n} + 2(-1)e^{-j\frac{\pi}{2}n} + e^{j\pi n}$$

(f) Extra Credit. for $n>0$ $(n\neq 0)$

$n<0$

$$h[n] = \frac{15}{8}\left(\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n\right)u[n]$$

$h[n]$ for n odd $\Rightarrow$ $= 0$

$g[n] = h[2n+1] = 0$ for all n

$E_g = 0$

Problem 2. (a)

(a) Consider $h_1[n]$ and $h_2[n]$ to be two distinct all-pass filters ($p_1 \neq p_2$) with respective impulse responses below. Is the sum $h[n] = h_1[n] + h_2[n]$ an all-pass filter for any and all values of p_1 and p_2 ($p_1 \neq p_2$) ?? Explain your answer. Your explanation is much more important than your answer.

$$h_1[n] = \frac{1}{p_1}\left\{\delta[n] \ + (p_1^2 - 1)p_1^n u[n]\right\} \tag{1}$$

$$h_2[n] = \frac{1}{p_2}\left\{\delta[n] \ + (p_2^2 - 1)p_2^n u[n]\right\} \tag{2}$$

$h[n] = h_1[n] + h_2[n] \Rightarrow$ parallel combination

Many past exams show that the parallel combination is either a notch-filter or digital resonator when $P_2 = -P_1 \Rightarrow$ NOT all-pass

Another argument:

$$H(\omega) = |H_1(\omega)| e^{j\angle H_1(\omega)} + |H_2(\omega)| e^{j\angle H_2(\omega)}$$

$$= 1 \cdot \underbrace{\left(e^{j\angle H_1(\omega)} + e^{j\angle H_2(\omega)}\right)}$$

since $|H_1(\omega)| = 1$, $|H_2(\omega)| = 2$ $\forall \omega$

will Have a magnitude that is NOT flat, in general for all frequencies

Problem 2.(b)

(b) Consider $h[n]$ to be an all-pass filter with respective impulse response below.

$$h[n] = \frac{1}{p}\left\{\delta[n] + (p^2 - 1)p^n u[n]\right\} \qquad \text{p is real-valued} \qquad (3)$$

Is the product

$$g[n] = e^{j\omega_o n} h[n]$$

an all-pass filter for any and all values of the frequency ω_o? Explain your answer. Your explanation is much more important than your answer.

From DTFT property: $G(\omega) = H(\omega - \omega_o)$

$$|G(\omega)| = |H(\omega - \omega_o)|$$

since $|H(\omega)| = 1$ for all ω $\quad -\infty < \omega < \infty$

$|H(\omega - \omega_o)| = 1$ for all ω $\Rightarrow$ $g(n)$ is all-pass

OR: can you result

$$r_{gg}[\ell] = e^{j\omega_o \ell} r_{hh}[\ell)$$

since $h(n)$ is all-pass

$$= e^{j\omega_o \cancel{\ell}} \delta[\ell]$$

$$= e^{j\omega_o (0)} \delta(\ell)$$

$$= \delta(\ell) \quad \Rightarrow \text{all-pass}$$

Problem 3. Note: this problem is different from a similar problem on last year's exam in that the two sequences, $x_1[n]$ and $x_2[n]$, are of different lengths.

(a) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-3 sequence $x_1[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_1[n] = \{1, 1, -1\} = \delta[n] + \delta[n-1] - \delta[n-2]$$

(b) Determine the autocorrelation $r_{x_2x_2}[\ell]$ of the length-5 sequence $x_2[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_2[n] = \{1, 1, 1, -1, 1\} = \delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4]$$

(c) The sequence $x_1[n]$ defined above is input to the system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_1x_1}[\ell]$ between the output and input.

$$y_1[n] = 4x_1[n-3] + x_1[n-4]$$

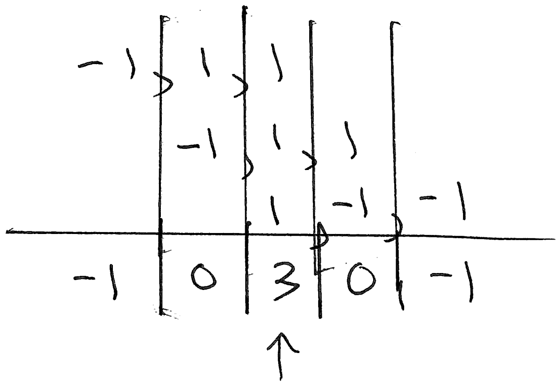
(d) The sequence $x_2[n]$ defined above is input to the same system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_2x_2}[\ell]$ between the output and input.

$$y_2[n] = 4x_2[n-3] + x_2[n-4]$$

(e) Sum your answers to parts (c) and (d) to form the sum below. Do a stem plot of $r_{yx}[\ell]$.

$$r_{yx}[\ell] = r_{y_1x_1}[\ell] + r_{y_2x_2}[\ell]$$

(a) $\{\underset{\uparrow}{1}, 1, -1\} * \{-1, 1, \underset{\uparrow}{1}\}$ ⇒ "Table Method" and know max value is at $\ell = 0$



$r_{x_1x_1}[\ell] = \{-1, 0, \underset{\uparrow}{3}, 0, -1\}$

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(b) $\{\underset{\uparrow}{1},1,1,-1,1\} * \{1,-1,1,1,\underset{\uparrow}{1}\}$

Table Method
max value is
at $\ell=0$

1	-1	1	1	1				
	1	-1	1	1	1			
		1	-1	1	1	1		
			-1	1	-1	-1	-1	
				1	-1	1	1	1
1	0	1	0	5	0	1	0	1

↑ $\ell=0$

$r_{x_2x_2}[\ell] =$

$\{1,0,1,0,\underset{\uparrow}{5},0,1,0,1\}$

$\ell=0$

(c) $r_{x_1x_1}[\ell] = 4r_{x_1x_1}[\ell-3] + r_{x_1x_1}[\ell-4] = h_1[\ell] * r_{x_1x_1}[\ell]$

$\{\underset{\uparrow}{0},0,0,4,1\} * \{-1,0,3,0,-1\}$

$\ell=0$

			-4	0	12	0	-4	
				-1	0	3	0	-1
0	0	0	-4	-1	12	3	-4	-1

$\{0,0,0,-4,-1,12,3,-4,-1\}$

^
|
ell=0

(d) $r_{Y_2X_2}[\ell] = h_2[\ell] * r_{X_2X_2}[\ell]$

$= 4\, r_{X_2X_2}[\ell-3] + r_{X_2X_2}[\ell-4]$

$\{\underset{\uparrow}{0}, 0, 0, 4, 1\} * \{1, 0, 1, 0, \underset{\uparrow}{5}, 0, 1, 0, 1\}$

$$\begin{array}{ccccccccccc}
4 & 0 & 4 & 0 & 20 & 6 & 4 & 0 & 4 & & \\
 & 1 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 1 & \\
\hline
\end{array}$$

$\{0, 0, 0, 4, 5, 4, 1, 20, 5, 4, 1, 4, 1\} = r_{Y_2X_2}[\ell]$

^
|
ell=0

(e) $r_{YX}[\ell] = r_{Y_1X_1}[\ell) + r_{Y_2X_2}[\ell)$

Due to linearity: define $r_{sum}[\ell] = r_{X_1X_1}[\ell) + r_{X_2X_2}[\ell)$

$= \{1, 0, 0, 0, \underset{\uparrow}{8}, 0, 0, 0, 1\}$

$r_{YX}[\ell] = \{\underset{\uparrow}{0}, 0, 0, 4, 1\} * \{1, 0, 0, 0, 8, 0, 0, 0, 1\}$

$$\begin{array}{ccc|c|c|c|c|c|c|c|c|c||c}
 & & & 4, & 0, & 0, & 0, & 32, & 0, & 0, & 0, & 4 & \\
 & & & & 1 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 4 & 1 & 0 & 0 & 32 & 8 & 0 & 0 & 4 & 1
\end{array}$$

^
|
ell=0

$\{0, 0, 0, 4, 1, 0, 0, 32, 8, 0, 0, 4, 1\}$ = answer

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