NAME: Solution to Exam 1

EE538 Digital Signal Processing I — Fall 2014

Exam 1 — Friday, Sept. 29, 2014

Cover Sheet

Write your name on this and every page
Test Duration: 60 minutes.
Coverage: Chapters 1-5.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **three** problems.
Show your work in the space provided for each problem.
You must show all work for each problem to receive full credit.
Always simplify your answers as much as possible.

Prob. No.	Topic(s)	Points
1.	Frequency Response and Interconnection of LTI Systems, Pole-Zero Diagrams	35
2.	DT Autocorrelation, Cross-Correlation and their Related Properties	30
3.	LTI Systems: Expressing Cross-Correlation in terms of Convolution	35

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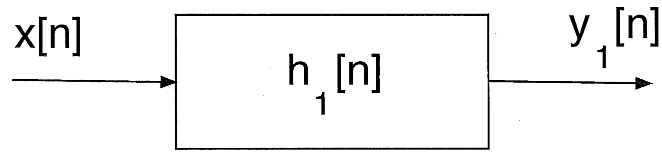
Problem 1. [35 points]

(a) Consider System 1 with impulse response $h_1[n]$ below, where $p_1 = 0.8$ (in fractional form $p_1 = \frac{4}{5}$):

$$\text{System 1:} \qquad h_1[n] = \frac{1}{p_1}\left\{\delta[n] + (p_1^2 - 1)p_1^n u[n]\right\}$$

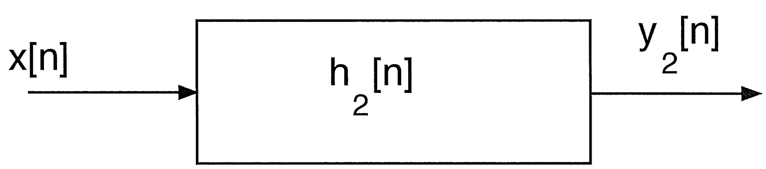


(i) Draw a pole-zero diagram for this system.

(ii) Plot the magnitude of the frequency response $|H_1(\omega)|$ over $-\pi < \omega < \pi$.

(b) Consider System 2 with impulse response $h_2[n]$ below, where $p_2 = -0.8$ (in fractional form $p_2 = -\frac{4}{5}$):

$$\text{System 2:} \qquad h_2[n] = \frac{-1}{p_2}\left\{\delta[n] + (p_2^2 - 1)p_2^n u[n]\right\}$$



(i) Draw a pole-zero diagram for this system.

(ii) Plot the magnitude of the frequency response $|H_2(\omega)|$ over $-\pi < \omega < \pi$.

(c) Consider Systems 1 and 2 to be connected in parallel. Plot the magnitude of the frequency response $|H(\omega)|$ of the parallel combination of System 1 and System 2 over $-\pi < \omega < \pi$.

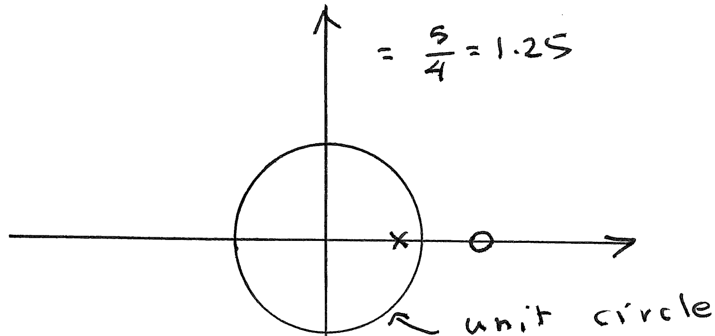
(d) Determine the overall output $y[n]$ of the parallel combination of System 1 and System 2 when the common input is:

$$x[n] = 3 + 2\cos(\pi n)$$

1 (a) $$H_1(z) = \frac{1}{p_1}\left\{1 + (p_1^2-1)\frac{z}{z-p_1}\right\}$$

$$= \frac{1}{p_1}\left\{\frac{z - p_1 + p_1^2 z - z}{z-p_1}\right\} = \frac{-1+p_1 z}{z-p_1}$$

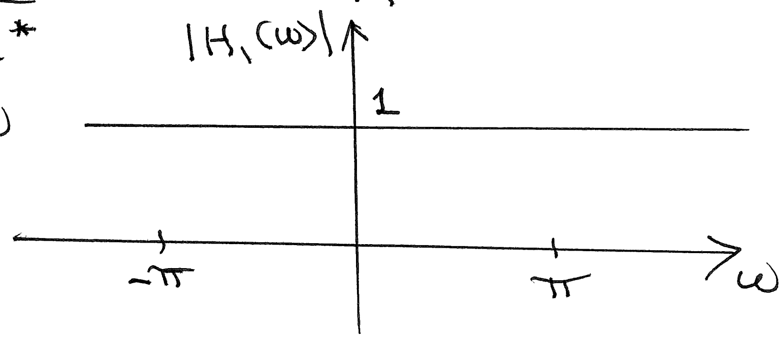
$p_1 = \frac{4}{5}$ zero at $z = 1/p_1 = \frac{5}{4} = 1.25$ pole at $z = p_1 = \frac{4}{5}$



$$H_1(\omega) = \frac{-1+p_1 e^{j\omega}}{e^{j\omega}-p_1} = -e^{j\omega}\frac{(e^{-j\omega}-p_1)}{e^{j\omega}-p_1}$$

$$= -e^{j\omega}\frac{c}{c^*}$$ since p_1 is real-valued

$$|H_1(\omega)| = 1 \quad \forall \omega$$



1 (b) $$H_2(z) = \frac{-1}{p_2}\left\{1 + (p_2^2-1)\frac{z}{z-p_2}\right\} =$$

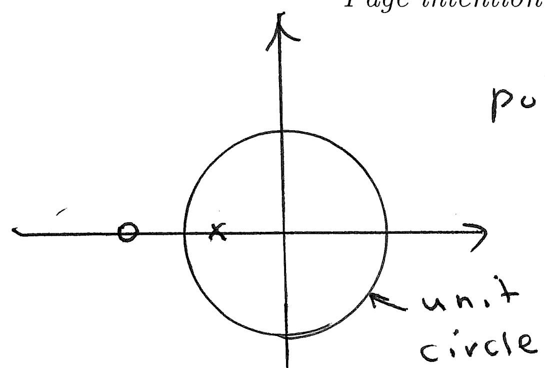
$$= -\frac{1}{p_2}\left\{\frac{z-p_2+p_2^2 z - z}{z-p_2}\right\} = -\frac{(-1+p_2 z)}{z-p_2}$$

zero at $z = 1/p_2$

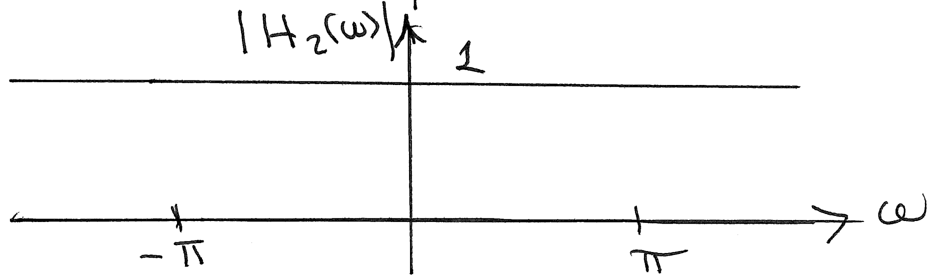
$p_2 = -\frac{4}{5}$ pole at $z = p_2$

1 (b) (cont.)

pole-zero diagram



$\Rightarrow$ all-pass



1 (c) $H(z) = H_1(z) + H_2(z)$

First, substitute $P_2 = -P_1$ $\Rightarrow$ and just set $P_1 = p$

$$\frac{-1+pz}{z-p} - \frac{(-1-pz)}{z+p} = \frac{-1+pz}{z-p} + \frac{1+pz}{z+p}$$

$$= \frac{(-1+pz)(z+p) + (1+pz)(z-p)}{(z-p)(z+p)}$$

$$= \frac{-z-p+pz^2+p^2z+z-p+pz^2-p^2z}{(z-p)(z+p)}$$

$$= \frac{-2p+2pz^2}{(z-p)(z+p)} = \frac{2p(z^2-1)}{(z-p)(z+p)}$$

zeroes at $z=1$ and $z=-1$ $\Rightarrow$ on unit circle
notch at $\omega = 0$ and $\omega = \pi$

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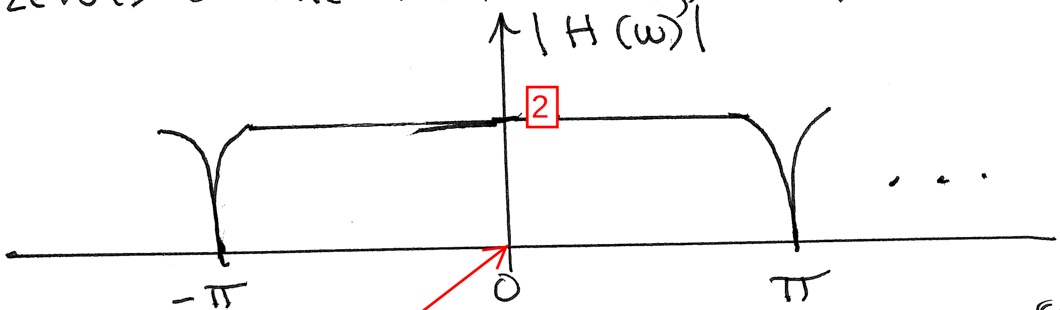
1(c) (cont.) since poles are at same angles as the zeroes on the unit circle ⇒ sharp notches



should be notch at w=0

$\cos(\pi n) = (-1)^n$

1(d) $x(n) = e^{j0\cdot n} + e^{j\pi n}$

$y(n) = \boxed{3}\, H(0)\, e^{j0\cdot n} + \boxed{2}\, H(\pi)\, e^{j\pi n}$

$= 0 + 0$

$= 0$ ⇒ both frequencies are notched out

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Problem 2. [30 points]

(a) Determine the autocorrelation $r_{x_0x_0}[\ell]$ of the length-4 sequence $x_0[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_0[n] = u[n] - u[n-4] = \{1, 1, 1, 1\}$$

(b) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-4 sequence $x_1[n]$ below. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_1[n] = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{\sqrt{2}}\right)} \{u[n] - u[n-4]\}$$

(c) Determine the autocorrelation $r_{x_2x_2}[\ell]$ of the length-4 sequence $x_2[n]$ below. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_2[n] = e^{j\pi(n-2)} \{u[n-2] - u[n-6]\}$$

(d) Determine the autocorrelation $r_{x_3x_3}[\ell]$ of the length-4 sequence $x_3[n]$ below. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_3[n] = e^{-j\left(\frac{\pi}{2}n + \frac{\pi}{\sqrt{2}}\right)} \{u[n] - u[n-4]\}$$

(e) Sum your answers to parts (a) thru (d) to form the sum below. Do a stem plot of $r_{xx}[\ell]$.

$$r_{xx}[\ell] = r_{x_0x_0}[\ell] + r_{x_1x_1}[\ell] + r_{x_2x_2}[\ell] + r_{x_3x_3}[\ell]$$

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2 (a) $x_0[n] = \{1, 1, 1, 1\}$

$$r_{x_0x_0}[\ell] = x_0[\ell] * x_0^*[-\ell] = x_0[\ell] * x_0[-\ell]$$

$$= x_0[\ell] * x_0[\ell+3]$$

1	1	1	1	0	0	0
0	1	1	1	1	0	0
0	0	1	1	1	1	0
0	0	0	1	1	1	1
1	2	3	4	3	2	1

sequence of 1's turned around is still a seq. of 1's

just do conv. with $x_0[\ell]$ and shift result to left by 3

$r_{x_0x_0}[\ell]$ max occurs at $\ell = 0$

$$r_{x_0x_0}[\ell] = \{1, 2, 3, \underset{\uparrow \ell=0}{4}, 3, 2, 1\}$$

2 (b) $x_1[n] = e^{j(\omega_0 n + 0)} x_0[n]$ $\quad \omega_0 = \frac{\pi}{2}$

$$r_{x_1x_1}[\ell] = e^{j\omega_0 \ell} r_{x_0x_0}[\ell] = (j)^\ell r_{x_0x_0}[\ell]$$

since $e^{j\frac{\pi}{2}\ell} = j^\ell$ $\quad j^0 = 1,\ j^2 = -1,\ j^1 = j$

$j^3 = -j$

$$\boxed{\begin{array}{l} j^0 = 1 \\ j^1 = j \\ j^2 = -1 \\ j^3 = -j \end{array}}$$

$$r_{x_1x_1}[\ell] = \{j, -2, -3j, \underset{\uparrow \ell=0}{4}, 3j, -2, -j\}$$

note: $r_{x_1x_1}[-\ell] = r^*_{x_1x_1}[\ell]$

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2 (c) $x_2[n] = x[n-2]$ <u>where</u>: $x[n] = e^{j\pi n} x_0[n]$

time-shift does not affect autocorrelation

Thus: $r_{x_2x_2}[\ell] = e^{j\pi \ell} r_{x_0x_0}[\ell] = (-1)^{\ell} r_{x_0x_0}[\ell]$

$$r_{x_2x_2}[\ell] = \{-1, 2, -3, \underset{\substack{\uparrow \\ \ell=0}}{4}, -3, 2, -1\}$$

2 (d) $x_3[n] = x_1^*(n) \Rightarrow r_{x_3x_3}[\ell] = r_{x_1x_1}^*[\ell]$

as proved in class during lecture right before exam

$$r_{x_3x_3}[\ell] = \{-j, -2, 3j, \underset{\substack{\uparrow \\ \ell=0}}{4}, -3j, -2, j\}$$

2(e) note: $r_{x_1x_1}[\ell] + r_{x_3x_3}[\ell] = 2\,\mathrm{Re}\{r_{x_1x_1}[\ell]\}$

Sum: =	0	-4	0	8	0	-4	0
$r_{x_0x_0}[\ell] \Rightarrow$	1	2	3	4	3	2	1
$r_{x_2x_2}[\ell] \Rightarrow$	-1	2	-3	4	-3	2	-1
	0	0	0	16	0	0	0

$r_{xx}[\ell] = 16\delta[\ell]$

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Problem 3. [35 points]

(a) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-4 sequence $x_1[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_1[n] = \{1, -1, 1, 1\} = \delta[n] - \delta[n-1] + \delta[n-2] + \delta[n-3]$$

(b) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-4 sequence $x_1[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_2[n] = \{-1, 1, 1, 1\} = -\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

(c) The sequence $x_1[n]$ defined above is input to the system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_1x_1}[\ell]$ between the output and input.

$$y_1[n] = x_1[n-4] + x_1[n-6]$$

(d) The sequence $x_2[n]$ defined above is input to the same system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_2x_2}[\ell]$ between the output and input.

$$y_2[n] = x_2[n-4] + x_2[n-6]$$

(e) Sum your answers to parts (c) thru (d) to form the sum below. Do a stem plot of $r_{yx}[\ell]$.

$$r_{yx}[\ell] = r_{y_1x_1}[\ell] + r_{y_2x_2}[\ell]$$

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3 (a) $x_1[n] = \{1, -1, 1, 1\}$

we know max of $r_{x_1x_1}[\ell]$ is at $\ell = 0$

$\{1, -1, 1, 1\} * \{1, 1, -1, 1\} = x_1[\ell] * x_1[-\ell]$

$$
\begin{array}{ccccccc}
1 & 1 & -1 & 1 & & & \\
 & -1 & -1 & 1 & -1 & & \\
 & & 1 & 1 & -1 & 1 & \\
 & & & 1 & 1 & -1 & 1 \\
\hline
\end{array}
$$

$r_{x_1x_1}[\ell]\{1, 0, -1, \underset{\uparrow}{4}, -1, 0, 1\}$

$\ell = 0$

3 (b) $x_2[n]$

$x_2[n] = \{-1, 1, 1, 1\}$

$\{-1, 1, 1, 1\} * \{1, 1, 1, -1\}$

and max of $r_{x_2x_2}[\ell]$ is at $\ell = 0$

$$
\begin{array}{ccccccc}
-1 & -1 & -1 & 1 & & & \\
 & 1 & 1 & 1 & -1 & & \\
 & & 1 & 1 & 1 & -1 & \\
 & & & 1 & 1 & 1 & -1 \\
\hline
\end{array}
$$

$\{-1, 0, 1, 4, 1, 0, -1\} = r_{x_2x_2}[\ell]$

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3(c) $y[n] = x[n-4] + x[n-6]$

$h[n] = \delta[n-4] + \delta[n-6]$

$r_{yx}[\ell] = h[\ell] * r_{xx}[\ell]$

$r_{y_1x_1}[\ell] = \{\delta[n-4] + \delta[n-6]\} * r_{x_1x_1}[\ell]$

$= r_{x_1x_1}[\ell-4] + r_{x_1x_1}[\ell-6]$

see plot on next page

3(d) Similarly,

$r_{y_2x_2}[\ell] = r_{x_2x_2}[\ell-4] + r_{x_2x_2}[\ell-6]$

see plot on next page

3(e) $r_{yx}[\ell] = r_{y_1x_1}[\ell] + r_{y_2x_2}[\ell]$

$= r_{x_1x_1}[\ell-4] + r_{x_2x_2}[\ell-4]$

$+ r_{x_1x_1}[\ell-6] + r_{x_2x_2}[\ell-6]$

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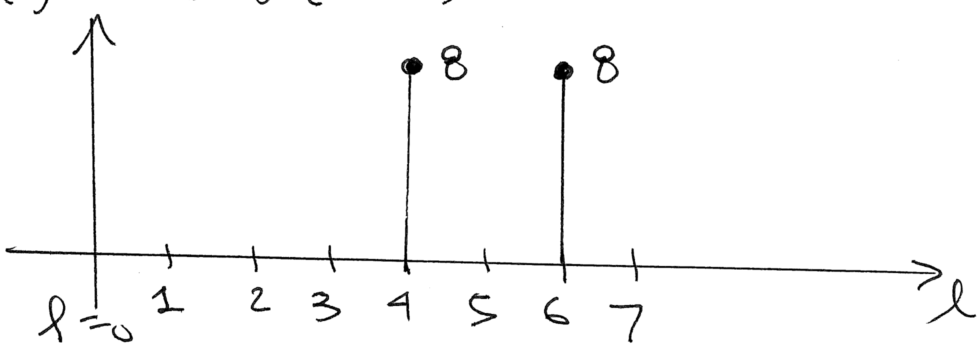
Denote: $r_{zz}[\ell] = r_{x_1x_1}[\ell] + r_{x_2x_2}[\ell]$

$= 8\,\delta[\ell]$ $\Rightarrow$ complementary length-4 Barker codes.

Thus:

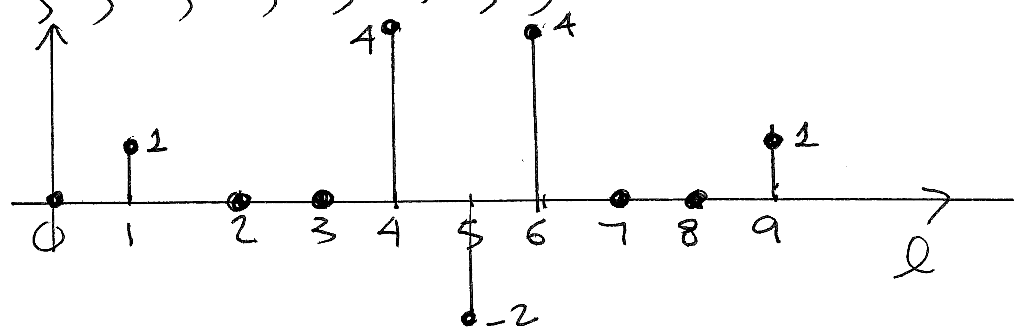
$r_{yx}[\ell] = 8\,\delta[\ell-4] + 8\,\delta[\ell-6]$



$r_{y_1x_1}[\ell] = \{0, 1, 0, 0, 4, -2, 4, 0, 0, 1\}$

3(c)

$\ell=0$



$r_{y_2x_2}[\ell] = \{0, -1, 0\ 0, 4, 2, 4, 0, 0, -1\}$

$\ell=0$