NAME: KEY / SOLUTION

EE538 Digital Signal Processing I **Fall 2013**
Exam 1 **Friday, Sept. 27, 2013**

Cover Sheet

Write your name on this and every page
Test Duration: 60 minutes.
Coverage: Chapters 1-5.
Open Book but Closed Notes.
Calculators NOT allowed.
This test contains **TWO** problems.
Show your work in the space provided for each problem.
You must show all work for each problem to receive full credit.
Always simplify your answers as much as possible.

Prob. No.	**Topic(s)**	**Points**
1.	Frequency Response and Interconnection of LTI Systems, Pole-Zero Diagrams	50
2.	DT Autocorrelation, Cross-Correlation Correlation in terms of Convolution	50

$$\frac{-.9j - z^{-1}}{1 - .9j z^{-1}} + \frac{.9j - z^{-1}}{1 + .9j z^{-1}}$$

$$\frac{-.9j z - 1}{z - .9j} \quad \frac{.9j z - 1}{z + .9j}$$

$$\frac{p_2^* z - 1}{z - p_2}$$

$(z - .9j)(z + .9j)$

$z^2 + .81$

$(-.9j z - 1)($

$p_2 = -p_1$

$$\frac{p^* z - 1}{z - p} + \frac{-p^* z - 1}{z + p}$$

$-0.9j$

$(p^* z - 1)(z + p) + (p^* z - 1)(z - p)$

$0.9j\left(1 - \frac{1}{0.9j}\right)$

$2 p^* z^2$

$(-.9j z - 1)(z + .9j) + (.9j z - 1)(z - .9j)$

$-.9j z^2 + .9j z^2 - z - z \quad -.9j \quad + .9j$

zero at $z=0$

$\underline{-2z}$

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Problem 1. [50 points] System 1 and System 2 defined in parts (a) and (b), respectively, will eventually be connected in parallel. We first analyze them individually.

(a) Consider System 1 below:

$$\text{System 1:} \quad y_1[n] = 0.9j\, y[n-1] - 0.9j\, x[n] - x[n-1]$$

(i) Determine the Transfer Function for System 1, denoted $H_1(z)$. $H_1(z)$ is the Z-Transform of the impulse response, $h_1[n]$, for System 1, although you can find $H_1(z)$ anyway you like. Identify the poles and zeros, and do a pole-zero plot.

(ii) Determine the frequency response of System 1, denoted $H_1(\omega)$. $H_1(\omega)$ is the DTFT of the impulse response, $h_1[n]$, for System 1, although you can find $H_1(\omega)$ anyway you like. Plot the magnitude $|H_1(\omega)|$ over $-\pi < \omega < \pi$.

(iii) Determine the autocorrelation of the impulse response $h_1[n]$. Do a stem plot of $r_{h_1 h_1}[\ell]$.

(b) Consider System 2 below:

$$\text{System 2:} \quad y_1[n] = -0.9j\, y[n-1] + 0.9j\, x[n] - x[n-1]$$

(i) Determine the Transfer Function for System 2, denoted $H_2(z)$. $H_2(z)$ is the Z-Transform of the impulse response, $h_2[n]$, for System 2, although you can find $H_2(z)$ anyway you like. Identify the poles and zeros, and do a pole-zero plot.

(ii) Determine the frequency response of System 2, denoted $H_2(\omega)$. $H_2(\omega)$ is the DTFT of the impulse response, $h_2[n]$, for System 2, although you can find $H_2(\omega)$ anyway you like. Plot the magnitude $|H_2(\omega)|$ over $-\pi < \omega < \pi$.

(iii) Determine the autocorrelation of the impulse response $h_2[n]$. Do a stem plot of $r_{h_2 h_2}[\ell]$.

(c) The overall system is formed from connecting System 1 and System 2 in parallel.

(i) Determine the Transfer Function for the overall system, denoted $H(z)$. $H(z)$ is the Z-Transform of the impulse response, $h[n]$, for the parallel combination of Systems 1 and 2. You can find $H(z)$ anyway you like. Identify the poles and zeros for the overall system, and do a pole-zero plot.

(ii) Determine the frequency response of the overall system, denoted $H(\omega)$. $H(\omega)$ is the DTFT of the impulse response, $h[n]$, for the overall system; you can find $H(\omega)$ anyway you like. Plot the magnitude $|H(\omega)|$ over $-\pi < \omega < \pi$. showing as much detail as possible. Point out any frequencies for which $H(\omega) = 0$.

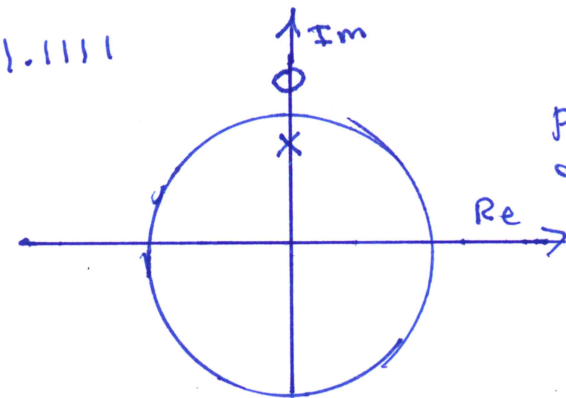
(iii) Determine the output of the overall system, $y[n]$, when the input is the DT signal below. The overall system is the parallel combination of Systems 1 and 2.

$$x[n] = 3 + 2\cos\left(\frac{\pi}{2}n\right) + (-1)^n \qquad -\infty < n < \infty$$

1(a)

(i) $H_1(z) = \dfrac{-.9j - z^{-1}}{1 - .9j\,z^{-1}} = \dfrac{-.9j\,z - 1}{z - .9j} = \dfrac{-.9j\left(z - \frac{1}{-.9j}\right)}{z - .9j}$

$.9 = \frac{9}{10} \Rightarrow \frac{1}{.9} = \frac{10}{9} = 1.1111$



pole-zero diagram for all-pass filter $H_1(z)$

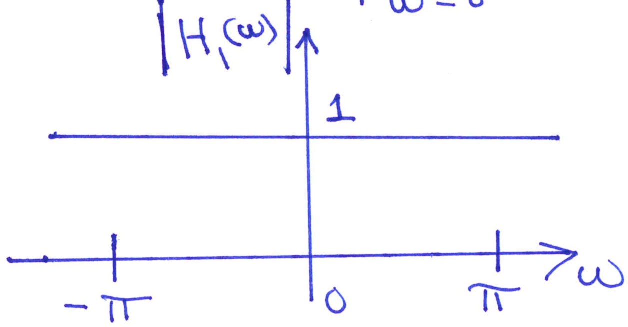
proved in class that $\dfrac{z - \frac{1}{p^*}}{z - p}$ is all-pass $\Rightarrow |H(\omega)| =$ constant $\forall \omega$

$\boxed{\left|\frac{c}{c^*}\right| = 1}$

$H_1(\omega)\Big|_{\omega=0} = H_1(z)\Big|_{z=1} = \dfrac{-.9j - 1}{1 - .9j} = \dfrac{-(1+.9j)}{1-.9j}$

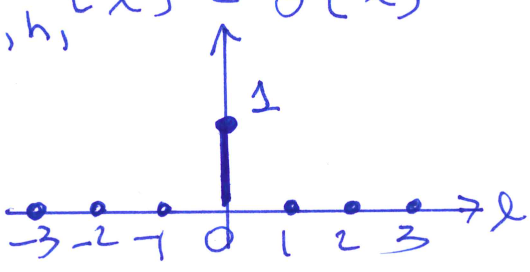
$\Rightarrow \left|H_1(\omega)\right|\Big|_{\omega=0} = 1 \Rightarrow |H_1(\omega)| = 1 \ \forall \omega$

(ii)



(iii) $r_{h_1h_1}[\ell] = h_1[\ell] * h_1^*[-\ell] \overset{DTFT}{\longleftrightarrow} H_1(\omega)H_1^*(\omega) = |H_1(\omega)|^2$

$r_{h_1h_1}[\ell] = \delta[\ell] \overset{DTFT}{\longleftrightarrow}$ since $|H_1(\omega)|^2 = 1$



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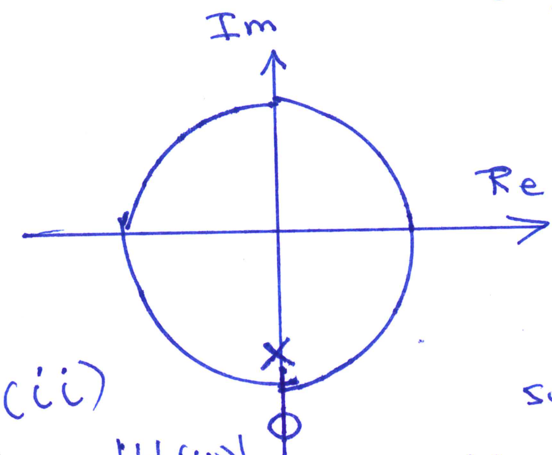
(b) $H_2(z) = \dfrac{.9j - z^{-1}}{1 + .9j z^{-1}} = \dfrac{.9j\left(z + \frac{1}{-.9j}\right)}{z + .9j} = \dfrac{.9j\left(z - \frac{1}{.9j}\right)}{z - (-.9j)}$

$= .9j \dfrac{\left(z - \frac{1}{p^*}\right)}{z - p} \Rightarrow$ all-pass filter

zero: $1/.9j = -j \frac{1}{.9}$

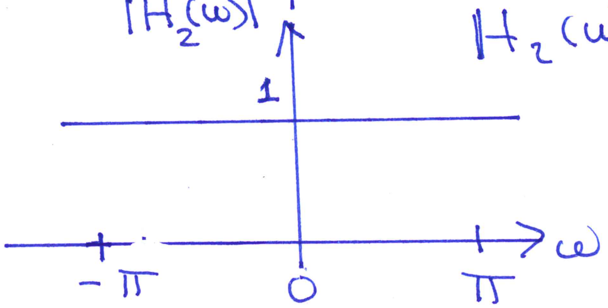
pole: $-.9j$



$|H_2(\omega)| = \text{constant} \ \forall \omega$

so, let's check $\omega = 0$ for $|H(\omega)|$

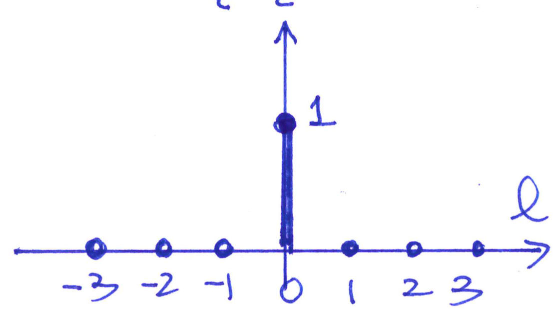
(ii)



$|H_2(\omega)|_{\omega=0} = \dfrac{.9j - 1}{1 + .9j} = \dfrac{-(1 - .9j)}{1 + .9j}$

$\Rightarrow |H_2(\omega)| = 1 \ \forall \omega$

(iii) Since $|H_2(\omega)|^2 = 1 \ \forall \omega \Rightarrow r_{h_2 h_2}[\ell] = \delta[\ell]$



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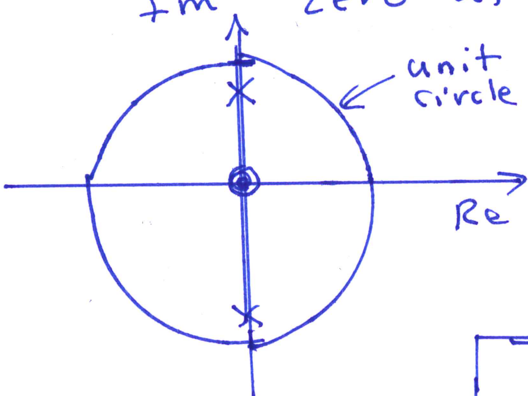
(c) $H(z) = H_1(z) + H_2(z)$

$$\frac{-.9jz - 1}{z - .9j} + \frac{.9jz - 1}{z + .9j} = \frac{(-.9jz-1)(z+.9j) + (.9jz-1)(z-.9j)}{(z-.9j)(z+.9j)}$$

$$= \frac{-.9jz^2 - z + .81z - .9j + .9jz^2 - z + .81z + .9j}{(z^2 + .81)}$$

$$= \frac{-2z + 2(.81)z}{z^2 + .81} = \frac{-0.38z}{z^2 + .81}$$

zero at $z = 0$ poles at $.9j$, $-.9j$



$\Rightarrow$ Digital Sinusoidal ~~Oscillator~~ Resonator

$\omega = 0$ $\quad H(0) = \dfrac{-0.38}{1 + .81} \approx -.21$

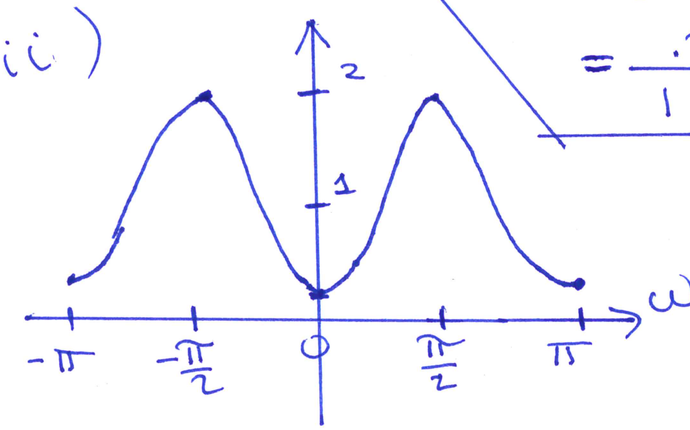
$\omega = \frac{\pi}{2}$ $\quad H(\frac{\pi}{2}) = \dfrac{-0.38j}{j^2 + .81} = \dfrac{-0.38j}{-1 + .81}$

$$= \frac{-0.38j}{-.19} = 2j$$

$\omega = \pi$ $\quad \boxed{H(\pi) = \dfrac{-.38(-1)}{(-1)^2 + .81} = \dfrac{.38}{1 + .81}}$

$|H(0)| = |H(\pi)| \approx 0.21$

(ii)



(iii) $x[n] = 3 + e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n} + e^{j\pi n}$

$y[n] = 3(-.21) + 2j e^{j\frac{\pi}{2}n}$

$+ 0.21 e^{j\pi n} - 2j e^{-j\frac{\pi}{2}n}$

$= 3(-0.21) + 0.21(-1)^n - 4 \boxed{\sin}\left(\frac{\pi}{2}n\right)$

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Problem 2. [50 points] Let $x[n]$ and $y[n]$ be DT signals with autocorrelations and cross-correlations defined in terms of convolution as below.

$$r_{xx}[\ell] = x[\ell] * x^*[-\ell] \quad r_{yy}[\ell] = y[\ell] * y^*[-\ell] \quad r_{xy}[\ell] = x[\ell] * y^*[-\ell] \quad r_{yx}[\ell] = y[\ell] * x^*[-\ell] \quad (1)$$

(a) Consider the case where $x[n]$ and $y[n]$ are both causal, finite-length signals of duration N. That is, $x[n]$ and $y[n]$ are both only nonzero for $n = 0, 1, ..., N-1$. A concatenated signal of length $2N$ is formed as below:

$$z[n] = x[n] + y[n-N] \quad (2)$$

Express the autocorrelation, $r_{zz}[\ell]$, for $z[n] = x[n] + y[n-N]$ in terms of $r_{xx}[\ell]$, $r_{yy}[\ell]$, $r_{xy}[\ell]$, and $r_{yx}[\ell]$.

(b) Now, consider the specific case where $x[n]$ and $y[n]$ form a complementary pair of +1's and -1's such that

$$r_{xx}[\ell] + r_{yy}[\ell] = 2N\delta[\ell] \quad (3)$$

Simplify your answer for $r_{zz}[\ell]$ in part (a) for this special case.

(c) Specifically, consider the Barker codes of length 4

$$x[n] = \{1, 1, 1, -1\} \qquad y[n] = \{1, 1, -1, 1\} \quad (4)$$

such that $z[n] = x[n] + y[n-N]$ is:

$$z[n] = \{1, 1, 1, -1, 1, 1, -1, 1\} \quad (5)$$

Determine the autocorrelation $r_{zz}[\ell]$ for $z[n]$ using the results that you derived above (you can compare to a direct calculation of $r_{zz}[\ell]$ to check your answer.)

(d) Now, we repeat the steps above for the case where the concatenated signal of length $2N$ is formed as below with a negative sign on the second term.

$$w[n] = x[n] - y[n-N] \quad (6)$$

Express the autocorrelation, $r_{ww}[\ell]$, for $w[n] = x[n] - y[n-N]$ in terms of $r_{xx}[\ell]$, $r_{yy}[\ell]$, $r_{xy}[\ell]$, and $r_{yx}[\ell]$.

(e) Again, consider the specific case where $x[n]$ and $y[n]$ form a complementary pair of +1's and -1's such that

$$r_{xx}[\ell] + r_{yy}[\ell] = 2N\delta[\ell] \quad (7)$$

Simplify your answer for $r_{ww}[\ell]$ in part (d) for this special case.

(f) Again, let $x[n] = \{1, 1, 1, -1\}$ and $y[n] = \{1, 1, -1, 1\}$ be Barker codes of length 4, such that

$$w[n] = x[n] - y[n-N] = \{1, 1, 1, -1, -1, -1, 1, -1\} \quad (8)$$

Determine the autocorrelation $r_{ww}[\ell]$ for $z[n]$ using the results that you derived above

(g) Sum your answers to parts (b) and (d) to form $r_{vv}[\ell] = r_{zz}[\ell] + r_{ww}[\ell]$, Do a stem plot of the sum $r_{vv}[\ell]$. Compare against the sum of your answers to (c) and (e).

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2(a) $r_{zz}[\ell] = \left(x[\ell] + y[\ell-N]\right) \underset{\wedge}{\overset{\text{convolution}}{*}} \left(x^*[-\ell] + y^*[-\ell-N]\right)$

$= x[\ell] * x^*[-\ell] + y[\ell-N] * y^*[-\ell-N]$

$+ y[\ell-N] * x^*[-\ell] + x[\ell] * y^*[-\ell-N]$

$= r_{xx}[\ell] + r_{yy}[\ell] + \delta[\ell-N] * y[\ell] * x^*[-\ell]$

$+ x[\ell] * y^*[-\ell] * \delta[\ell+N]$

since: $y[-\ell-N] = y[-(\ell+N)] = y[-\ell] * \delta[\ell+N]$

<u>Thus</u>: $r_{zz}[\ell] = r_{xx}[\ell] + r_{yy}[\ell] + r_{yx}[\ell-N] + r_{xy}[\ell+N]$

(b) $r_{zz}[\ell] = 2N\delta[\ell] + r_{yx}[\ell-N] + r_{xy}[\ell+N]$

(c) $x[n]$ and $y[n]$ are a complementary pair

Thus, we just need to find $r_{xy}[\ell]$ and $r_{yx}[\ell]$

$r_{yx}[\ell] = y[\ell] * x^*[-\ell] = \{-1, 0, 3, \underset{\substack{\uparrow \\ \ell=0}}{0}, 1, 0, 1\}$

$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \{1, 0, 1, \underset{\substack{\uparrow \\ \ell=0}}{0}, 3, 0, -1\}$

Thus,

$r_{zz}[\ell] = 2(4)\delta[\ell] + r_{yx}[\ell-4] + r_{xy}[\ell+4]$

$= \{1, 0, 1, 0, 3, 0, -1, \underset{\uparrow}{8}, -1, 0, 3, 0, 1, 0, 1\}$

(d) $w[n] = x[n] - y[n-N]$

Easy to deduce from part (a)

$$r_{ww}[\ell] = r_{xx}[\ell] + r_{yy}[\ell] - r_{yx}[\ell-N] - r_{xy}[\ell+N]$$

(e) $r_{ww}[\ell] = 2N\,\delta[\ell] - r_{yx}[\ell-N] - r_{xy}[\ell+N]$

(f) $r_{ww}[\ell] = \{-1, 0, -1, 0, -3, 0, 1, \underset{\uparrow\,\ell=0}{8}, 1, 0, -3, 0, -1, 0, -1\}$

(g) $r_{vv}[\ell] = r_{zz}[\ell] + r_{ww}[\ell]$

$= r_{xx}[\ell] + r_{yy}[\ell] + r_{yx}[\ell-N] + r_{xy}[\ell+N]$

$+ r_{xx}[\ell] + r_{yy}[\ell] - r_{yx}[\ell-N] - r_{xy}[\ell-N]$

$= 2\left(r_{xx}[\ell] + r_{yy}[\ell]\right)$

$= 2\left(2N\,\delta[\ell]\right) = 4N\,\delta[\ell]$

$= 16\,\delta[\ell]$

Same answer using numbers above

$r_{zz}[\ell] = \{1, 0, 1, 0, 3, 0, -1, 8, -1, 0, 3, 0, 1, 0, 1\}$

$+ r_{ww}[\ell] = \{-1, 0, -1, 0, -3, 0, 1, 8, 1, 0, -3, 0, -1, 0, -1\}$

$\{\underset{\uparrow\,\ell=0}{16}\}$

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