NAME: Student Solution

EE538 Digital Signal Processing I — Fall 2018
Exam 1 — Friday, Sept. 28, 2018

Cover Sheet

Write your name on this and every page
Test Duration: 60 minutes.
Coverage: Chapters 1-5.
Open Book but Closed Notes.
Allowed one double-sided 8.5 x 11 handwritten or typed crib sheet
Calculators NOT allowed.
This test contains **TWO** problems.
Show your work in the space provided for each problem.
You must show all work for each problem to receive full credit.
Always simplify your answers as much as possible.

Prob. No.	**Topic(s)**	**Points**
1.	Frequency Response of LTI Systems Pole-Zero Diagrams DT Autocorrelation, Cross-Correlation and their Related Properties	60
2.	LTI Systems: Autocorrelation and Cross-Correlation and their Related Properties	40

Problem 1.[60 pts] Consider the DT System characterized by the difference equation

$$\text{System 1:} \quad y_1[n] = y_1[n-1] + x[n] - x[n-4] \tag{1}$$

(a) Draw the pole-zero diagram for this system.

(b) Plot the magnitude of the frequency response $|H_1(\omega)|$ over $-\pi < \omega < \pi$.

(c) Determine the overall output $y_1[n]$ of the System when the input is the sum of infinite-length sinewaves below

$$x[n] = 2 + e^{j\frac{\pi}{2}n} + \sqrt{2}e^{-j\frac{\pi}{2}n} + 3e^{j\pi n} \tag{2}$$

(d) Determine the impulse response of the system, $h[n]$. Write the values out in sequence form. *Hint:* the impulse response, $h_1[n]$, is finite-length.

(e) Determine the autocorrelation of the output, $r_{yy}[\ell] = y[\ell] * y^*[-\ell]$ with the input signal below, with $p = \frac{1}{2}$. Write the values out in sequence form; indicate $\ell = 0$ point.

$$x[n] = \frac{1}{p}\left\{\delta[n] + (p^2 - 1)p^n u[n]\right\} \tag{3}$$

(f) Repeat parts (a) thru (e) for System 2 below. Certainly make use of everything you did for System 1 since only the location of the pole has changed. Denote your answer for this part (f) as $r_{y_2y_2}[\ell] = y_2[\ell] * y_2^*[-\ell]$

$$\text{System 2:} \quad y_2[n] = j\, y_2[n-1] + x[n] - x[n-4] \tag{4}$$

Hint: You can determine the autocorrelation any way you want to, but it may be helpful to first write the impulse response in the form $h[n] = e^{j\omega_o n}\{u[n] - u[n-N]\}$ (determine ω_o and N) and then use a property of autocorrelation derived in class.

(g) Repeat parts (a) thru (e) for System 3 below. Certainly make use of everything you did for System 1. Denote your answer for this part (g) as $r_{y_3y_3}[\ell] = y_3[\ell] * y_3^*[-\ell]$

$$\text{System 3:} \quad y_3[n] = -\, y_3[n-1] + x[n] - x[n-4] \tag{5}$$

(h) Repeat parts (a) thru (e) for System 4 below. Certainly make use of everything you did for System 1. Denote your answer for this part (h) as $r_{y_4y_4}[\ell] = y_4[\ell] * y_4^*[-\ell]$

$$\text{System 4:} \quad y_4[n] = -j\, y_4[n-1] + x[n] - x[n-4] \tag{6}$$

(i) Sum your answers as indicated below. Write your final answer in sequence form.

$$r_{yy}[\ell] = r_{y_1y_1}[\ell] + r_{y_2y_2}[\ell] + r_{y_3y_3}[\ell] + r_{y_4y_4}[\ell] \tag{7}$$

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1) a) $Y_1(z) = z^{-1}Y_1(z) + X(z) - z^{-4}X(z)$

$$Y_1(z)(1-z^{-1}) = X(z)(1-z^{-4})$$

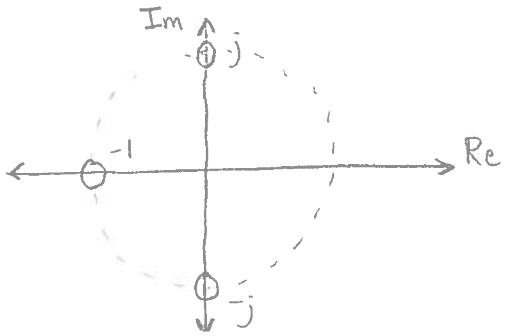
$$\frac{Y_1(z)}{X(z)} = H(z) = \frac{1-z^{-4}}{1-z^{-1}} = \frac{z^4-1}{z^3(z-1)}$$

Zeros: $z = \pm 1, \pm j$

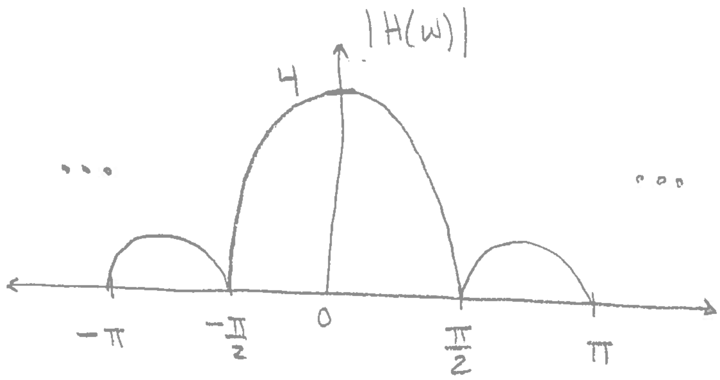
Poles: $z = 1$

Z-P cancellation @ $z=1$

plus 3 poles at z=0



b) $$H(\omega) = \frac{\sin\left(\frac{4}{2}(\omega)\right)}{\sin\left(\frac{1}{2}(\omega)\right)} e^{-j\frac{3}{2}\omega}$$



d) $h[n] = e^{j\omega_0 n}\{u[n] - u[n-4]\}$

$= e^{j(0)n}\{\underset{\uparrow}{1}, 1, 1, 1\} = \{\underset{\uparrow}{1}, 1, 1, 1\}$

c) $y_1[n] = H(0)\cdot 2 + H(\frac{\pi}{2})e^{j\frac{\pi}{2}n} + H(-\frac{\pi}{2})e^{-j\frac{\pi}{2}n} \cdot \sqrt{2} + H(\pi)\cdot 3e^{j\pi n}$

$= 4\cdot 2 + 0 + 0 + 0 = 8$

e) Proved in class that $r_{xx}[\ell] = \delta[\ell]$ for $-1 < p < 1$ $p = \frac{1}{2}$ ✓

$r_{yy}[\ell] = r_{hh}[\ell] * r_{xx}[\ell]$

$= \{1, 2, 3, \underset{\uparrow}{4}, 3, 2, 1\} * \delta[\ell] = \{1, 2, 3, \underset{\uparrow}{4}, 3, 2, 1\}$

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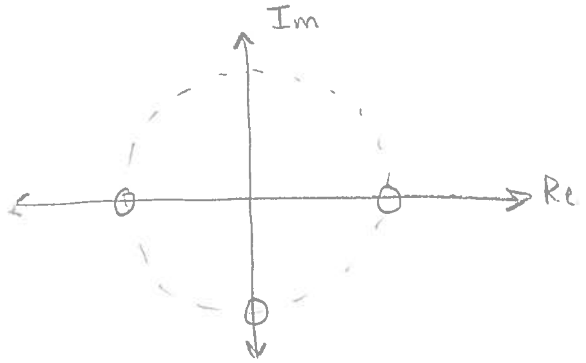
f) Same as part a) but pole is at $+j$

$$H(z) = \frac{z^4 - 1}{z - j}$$

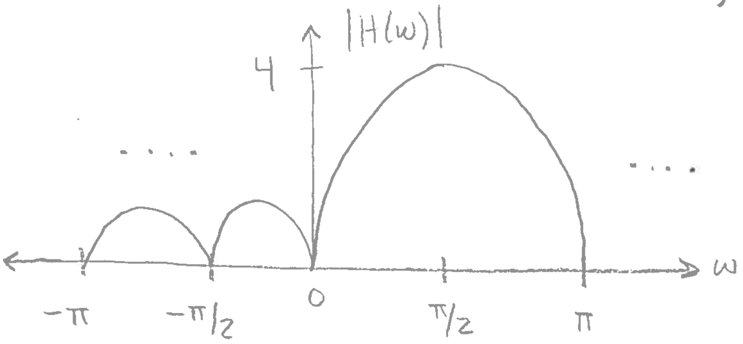
Zeros: $z = \pm 1, \pm j$

Poles: $z = j$

z-p cancellation @ $z = j$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}\left(\omega - \frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{\pi}{2}\right)\right)} e^{-j\frac{3}{2}\left(\omega - \frac{\pi}{2}\right)}$$



$$y_2[n] = H(0)\cdot 2 + H\left(\tfrac{\pi}{2}\right) e^{j\frac{\pi}{2}n} + H\left(-\tfrac{\pi}{2}\right)\cdot\sqrt{2}\, e^{-j\frac{\pi}{2}n} + H(\pi)\cdot 3 e^{j\pi n}$$

$$= 0 + 4e^{j\pi/2 n} + 0 + 0 = 4e^{j\frac{\pi}{2}n}$$

$$h_2[n] = e^{j\omega_0 n}\{\underset{\uparrow}{1}, 1, 1, 1\} = e^{j\frac{\pi}{2}n}\{\underset{\uparrow}{1}, 1, 1, 1\} = \{1, j, -1,$$

$$r_{y_2y_2}[\ell] = r_{h_2h_2}[\ell] * r_{x_2x_2}[\ell]$$

$$= \{-j\ \textcolor{red}{-2}, -3j, \underset{\uparrow}{4}, 3j, -2, \textcolor{red}{-j}\} * \delta[\ell]$$

$$= \{-j, 2, -3j, \underset{\uparrow}{4}, 3j, -2,$$

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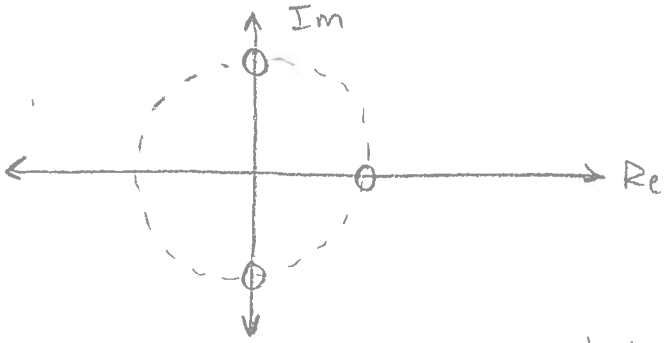
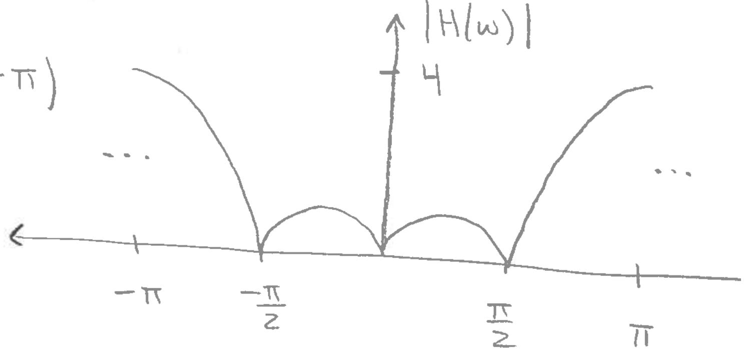
g) Same as a) but pole is at -1

$$H(z) = \frac{z^4 - 1}{z + 1}$$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}(\omega - \pi)\right)}{\sin\left(\frac{1}{2}(\omega - \pi)\right)} e^{-j\frac{3}{2}(\omega - \pi)}$$



$$y_3[n] = H(0)\cdot 2 + H(\tfrac{\pi}{2}) e^{j\frac{\pi}{2}n} + H(\tfrac{-\pi}{2})\cdot\sqrt{2}\, e^{-j\frac{\pi}{2}n} + H(\pi)\cdot 3 e^{j\pi n}$$

$$= 0 + 0 + 0 + 4\cdot 3 e^{j\pi n} = 12 e^{j\pi n}$$

$$h_3[n] = e^{j\omega_0 n}\{\underset{\uparrow}{1}, 1, 1, 1\} = e^{j\pi n}\{\underset{\uparrow}{1}, 1, 1, 1\} = \{1, -1, 1, -1$$

$$r_{y_3y_3}[\ell] = r_{h_3h_3}[\ell] * r_{x_3x_3}[\ell] = \{1, 2, 3, \underset{\uparrow}{4}, 3, 2, 1\} * \delta[$$

$$= \{-1, 2, -3, \underset{\uparrow}{4}, -3, 2, -1\}$$

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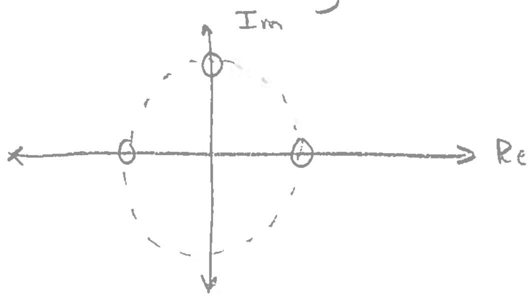
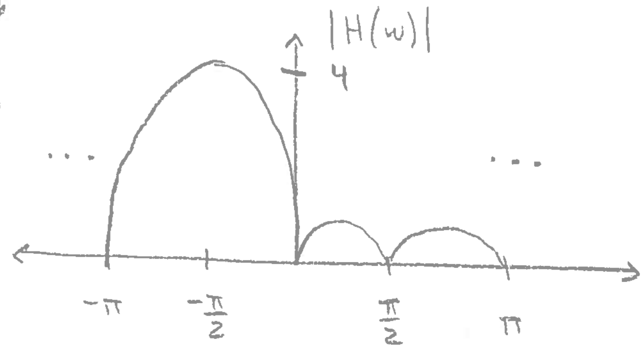
h) Same as a) but pole is at $-j$

$$H(z) = \frac{z^4 - 1}{z + j}$$



$$H(\omega) = \frac{\sin\left(\frac{4}{2}\left(\omega + \frac{\pi}{2}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega + \frac{\pi}{2}\right)\right)} e^{-j\frac{3}{2}\left(\omega + \frac{\pi}{2}\right)}$$



$$y_4[n] = H(0)\cdot 2 + H\left(\tfrac{\pi}{2}\right)\cdot e^{j\frac{\pi}{2}n} + H\left(\tfrac{-\pi}{2}\right)\cdot\sqrt{2}e^{-j\frac{\pi}{2}n} + H(\pi)\cdot 3e^{j\pi n}$$

$$= 0 + 0 + 4\cdot\sqrt{2}e^{-j\frac{\pi}{2}n} + 0 = 4\left(\sqrt{2}e^{-j\frac{\pi}{2}n}\right)$$

$$h_4[n] = e^{j\omega_0 n}\{\underset{\uparrow}{1}, 1, 1, 1\} = e^{j\frac{-\pi}{2}n}\{1, 1, 1, 1\} = \{1, -j, -1, j\}$$

$$r_{y_4y_4}[\ell] = r_{h_4h_4}[\ell] * r_{x_4x_4}[\ell] = \{-j, -2, 3j, \underset{\uparrow}{4}, -3j, -2, +j\}$$

1 -j -1 j

1 j -1

1 j

1

i) $r_{yy}[\ell] = r_{y_1y_1}[\ell] + r_{y_2y_2}[\ell] + r_{y_3y_3}[\ell] + r_{y_4y_4}[\ell]$

$= 4\delta[\ell] = 4(4) = 16$

$= \{0, 0, 0, \underset{\uparrow}{16}, 0, 0, 0\}$

Problem 2. [40 points]

(a) Determine the autocorrelation $r_{x_1x_1}[\ell]$ of the length-5 sequence $x_1[n]$ (Barker code) below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_1[n] = \{1, 1, 1, -1, 1\} = \delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-3] + \delta[n-4]$$

(b) Determine the autocorrelation $r_{x_2x_2}[\ell]$ of the length-7 sequence $x_2[n]$ (Barker code) below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$x_2[n] = \{1, 1, 1, 1, -1, 1, -1\} = \delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]-\delta[n-4]+\delta[n-5]-\delta[n-6]$$

(c) The sequence $x_1[n]$ defined above is input to the system described by the simple difference equation below. Determine the cross-correlation between the output and input, $r_{y_1x_1}[\ell]$; either write out the values in sequence form indicating $\ell = 0$ or do stem plot.

$$y_1[n] = 2x_1[n-3] - x_1[n-4]$$

(d) The sequence $x_2[n]$ defined above is input to the SAME system described by the simple difference equation below. Determine the cross-correlation between the output and input, $r_{y_2x_2}[\ell]$; either write out the values in sequence form indicating $\ell = 0$ or do stem plot.

$$y_2[n] = 2x_2[n-3] - x_2[n-4]$$

(e) Sum your answers to parts (c) and (d) to form the sum below. Write out in sequence form or do a stem plot of $r_{yx}[\ell]$.

$$r_{yx}[\ell] = r_{y_1x_1}[\ell] + r_{y_2x_2}[\ell]$$

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a)

$$\begin{array}{ccccc} 1 & 1 & 1 & -1 & 1 \\ 1 & & & & \\ -1 & 1 & & & \\ 1 & -1 & 1 & & \\ 1 & 1 & -1 & 1 & \\ 1 & 1 & 1 & -1 & 1 \end{array}$$

$$= \{1, 0, 1, 0, \underset{\uparrow \, \ell=0}{5}, 0, 1, 0, 1\}$$

b)

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & & & & & & \\ 1 & -1 & & & & & \\ -1 & 1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & \\ -1 & 1 & -1 & 1 & -1 & & \\ 1 & -1 & 1 & -1 & 1 & -1 & \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 \end{array}$$

The correct length=7 Barker sequence is:
x_2 [n]={ 1 1 1 -1 -1 1 -1}
There was a typo on this student's exam

$$= \{-1, 0, -1, 0, -1, 0, \underset{\uparrow \, \ell=0}{7}, 0, -1, 0, -1, 0, -1$$

term = +3?
know it's Barker code
so term must be 0, -1, 1
1 + 1 + (-1) + 1 + 1 = 3 ?

c) $r_{y_1 x_1}[\ell] = h_1[\ell] * r_{x_1 x_1}[\ell] = \{\underset{\uparrow}{0}, 0, 0, 2, -1\} * r_{x_1 x_1}[\ell]$

$= 2 r_{x_1 x_1}[\ell - 3] - r_{x_1 x_1}[\ell - 4]$

$$= \begin{array}{ccccccccccccc} & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 10 & 0 & 2 & 0 & 2 \\ - \; 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 1 \end{array}$$

$$= \{0, 0, 0, 2, \underset{\uparrow \, \ell=0}{-1}, 2, -1, 10, -5, 2, -1, 2, -1\}$$

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d) $r_{y_2x_2}[\ell] = h_2[\ell] * r_{x_2x_2}[\ell] = \{\underset{\uparrow \ell=0}{0}, 0, 0, 2, -1\} * r_{x_2x_2}[\ell$

$= 2r_{x_2x_2}[\ell-3] - r_{x_2x_2}[\ell-4]$

$= \quad 0 \quad 0 \quad 0 \quad -2 \quad 0 \quad -2 \quad 0 \quad 2 \quad 0 \quad 14 \quad 0 \quad 2 \quad 0 \quad -2 \quad 0 \quad -2$

$- \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad 7 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad -1$

$= \{0, 0, 0, -2, 1, -2, \underset{\uparrow \ell=0}{1}, 2, -1, 14, -7, 2, -1, -2, 1, -2, 1\}$

e) $r_{yx}[\ell] = r_{y_1x_1}[\ell] + r_{y_2x_2}[\ell]$

$= \quad 0 \quad 0 \quad 0 \quad 2 \quad -1 \quad 2 \quad -1 \quad 10 \quad -5 \quad 2 \quad -1 \quad 2 \quad -1$

$+ \quad 0 \quad 0 \quad 0 \quad -2 \quad 1 \quad -2 \quad 1 \quad 2 \quad -1 \quad 14 \quad -7 \quad 2 \quad -1 \quad -2 \quad 1 \quad -2 \quad 1$

$= \{0, 0, 0, -2, 1, 0, \underset{\uparrow \ell=0}{0}, 4, -2, 24, -12, 4, -2, 0, 0, \ldots, -2, 1$

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