Exam 1: Short Version to Keep When You Leave

Problem 1. [40 points]

(a) Note: System 1 and System 2 defined in parts (b) and (c), respectively, are NOT in parallel, but they do have the same input equal to $x[n]$ below with $p = \frac{3}{4}$.

$$x[n] = \frac{1}{p} \left\{ \delta[n] + (p^2 - 1)p^n u[n] \right\}$$  \hspace{1cm} (1)

Determine & plot the autocorrelation sequence $r_{xx}[\ell]$ for $x[n]$ defined above with $p = \frac{3}{4}$.

(b) Consider System 1 below:

**System 1:**

$$y_1[n] = x[n] + x[n-1] - x[n-2] + x[n-3]$$

(i) Determine the impulse response $h_1[n]$. You can write it in sequence form.

(ii) Determine the autocorrelation of the impulse response. Do a stem plot of $r_{h1h1}[\ell]$.

(iii) For the input signal in Equation (1) above with $p = \frac{3}{4}$, determine the autocorrelation of the output $y_1[n]$. Do a stem plot of $r_{y1y1}[\ell]$.

(c) Consider System 2 below:

**System 2:**

$$y_2[n] = x[n] + x[n-1] + x[n-2] - x[n-3]$$

(i) Determine the impulse response $h_2[n]$. You can write it in sequence form.

(ii) Determine the autocorrelation of the impulse response. Do a stem plot of $r_{h2h2}[\ell]$.

(iii) Determine the autocorrelation of the output $y_2[n]$. Do a stem plot of $r_{y2y2}[\ell]$.

(d) Using your answers from parts (b) and (c), do a stem plot of $r[\ell] = r_{y1y1}[\ell] + r_{y2y2}[\ell]$. 
Problem 2. [20 points] System 1 and System 2 are as defined in parts (b) and (c) of Problem 1, respectively, but now they are connected in parallel, such that the overall output is

\[ y[n] = y_1[n] + y_2[n] \]

(a) Determine the overall response \( h[n] \) of the system. Do a stem plot of \( h[n] \).

(b) Plot the magnitude of the frequency response of the system \( |H(\omega)| \) over \(-\pi < \omega < \pi\).

(c) Determine a closed-form expression for the output \( y[n] \) when the input is \( x[n] \) below.

\[ x[n] = 1 + (j)^n + (-j)^n + (-1)^n \]

Problem 3. [40 points]

(a) The autocorrelation of \( x[n] \) may be computed in terms of convolution as

\[ r_{xx}[\ell] = x[\ell] \ast x^*[-\ell] \]

Which of the three main properties of convolution discussed in class is key to proving that the autocorrelation of \( y[n] = x^*[-n] \) is the same as the autocorrelation of \( x[n] \). Write your one-word answer directly below.

(b) Is the convolution of two all-pass filters/sequences also an all-pass filter/sequence? Prove your answer with a short mathematical analysis in the space provided on the next few blank pages.

(c) Let \( x[n] \) be the input signal described below with \( p = \frac{1}{2} \).

\[ x[n] = \frac{1}{p} \left\{ \delta[n] + (p^2 - 1)p^n u[n] \right\} \]

This signal is input to the system below described the difference equation below

\[ y[n] = \frac{2}{3} y[n - 1] - \frac{2}{3} x[n] + x[n - 1] \]

(i) Draw the pole-zero diagram for this system. Is the system stable?

(ii) Determine the autocorrelation of the output \( y[n] \). Do a stem plot of \( r_{yy}[\ell] \).

(d) For the same input signal and same system described in part (c), determine the cross-correlation \( r_{yx}[\ell] \) between the input and output. Just provide a closed-form expression for \( r_{yx}[\ell] \); you don’t need to plot it.