NAME: KEY

EE538 Digital Signal Processing I — Fall 2012

Exam 1 — Friday, Sept. 28, 2012

Cover Sheet

Write your name on this and every page

Test Duration: 60 minutes.

Coverage: Chapters 1-5.

Open Book but Closed Notes.

Calculators NOT allowed.

This test contains **three** problems.

Show your work in the space provided for each problem.

You must show all work for each problem to receive full credit.

Always simplify your answers as much as possible.

Prob. No.	Topic(s)	Points
1.	DT Autocorrelation, Cross-Correlation Correlation in terms of Convolution	40
2.	Frequency Response and Interconnection of LTI Systems	20
3.	LTI Systems: Properties of Convolution Pole-Zero Diagrams	40

NAME: KEY

Problem 1. [40 points]

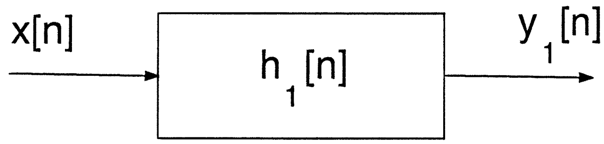
(a) **Note:** System 1 *and* System 2 defined in parts (b) and (c), respectively, are NOT in parallel, but they do have the same input equal to $x[n]$ below with $p = \frac{3}{4}$.

$$x[n] = \frac{1}{p}\left\{\delta[n] \; + (p^2 - 1)p^n u[n]\right\} \tag{1}$$

Determine & plot the autocorrelation sequence $r_{xx}[\ell]$ for $x[n]$ defined above with $p = \frac{3}{4}$.

(b) Consider System 1 below:

System 1: $\quad y_1[n] = x[n] + x[n-1] - x[n-2] + x[n-3]$



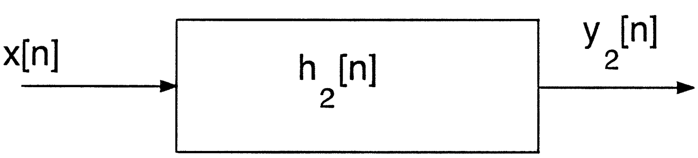
(i) Determine the impulse response $h_1[n]$. You can write it in sequence form.

(ii) Determine the autocorrelation of the impulse response. Do a stem plot of $r_{h_1h_1}[\ell]$.

(iii) For the input signal in Equation (1) above with $p = \frac{3}{4}$, determine the autocorrelation of the output $y_1[n]$. Do a stem plot of $r_{y_1y_1}[\ell]$.

(c) Consider System 2 below:

System 2: $\quad y_2[n] = x[n] + x[n-1] + x[n-2] - x[n-3]$



(i) Determine the impulse response $h_2[n]$. You can write it in sequence form.

(ii) Determine the autocorrelation of the impulse response. Do a stem plot of $r_{h_2h_2}[\ell]$.

(iii) Determine the autocorrelation of the output $y_2[n]$. Do a stem plot of $r_{y_2y_2}[\ell]$.

(d) Using your answers from parts (b) and (c), do a stem plot of $r[\ell] = r_{y_1y_1}[\ell] + r_{y_2y_2}[\ell]$.

NAME: KEY

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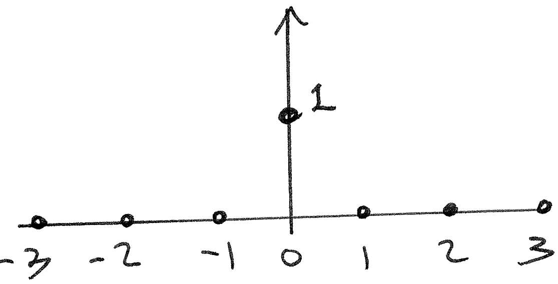
1 (a) $x[n] = \frac{1}{p}\left\{\delta[n] + (p^2-1)\,p^n u[n]\right\}$

$\Rightarrow$ all-pass signal with $p = 3/4$

$r_{xx}[\ell] = \delta[\ell]$



1 (b) $h_1[n] = \delta[n] + \delta[n-1] - \delta[n-2] + \delta[n-3]$

$= \{\underset{\uparrow}{1}, 1, -1, 1\}$

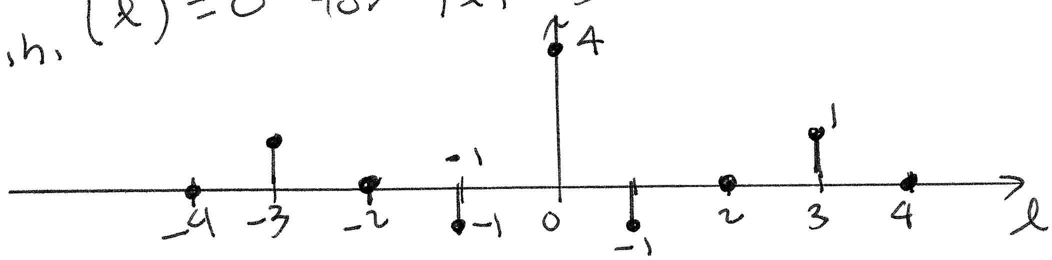
$r_{h_1h_1}[0] = 1^2 + 1^2 + (-1)^2 + (1)^2 = 4$

$r_{h_1h_1}[1] = \{1, 1, -1, 1\}$ $\{1, 1, -1, 1\}$ $= -1 = r_{h_1h_1}[-1]$

$r_{h_1h_1}[2] = \{1, 1, -1, 1\}$ $\{1, 1, -1, 1\}$ $= 0 = r_{h_1h_1}[-2]$

$r_{h_1h_1}[3] = \{1, 1, -1, 1\}$ $1, 1, -1, 1$ $= 1 = r_{h_1h_1}[-3]$

$r_{h_1h_1}[\ell] = 0$ for $|\ell| > 3$



$r_{y_1y_1}[\ell] = r_{h_1h_1}[\ell] * r_{xx}[\ell] = r_{h_1h_1}[\ell] * \delta[\ell] = r_{h_1h_1}[\ell]$

$\Rightarrow$ same plot as above

NAME: KEY

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1 (c) $h_2[n] = \delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-3]$

(i) $= \{\underset{\uparrow}{1}, 1, 1, -1\}$

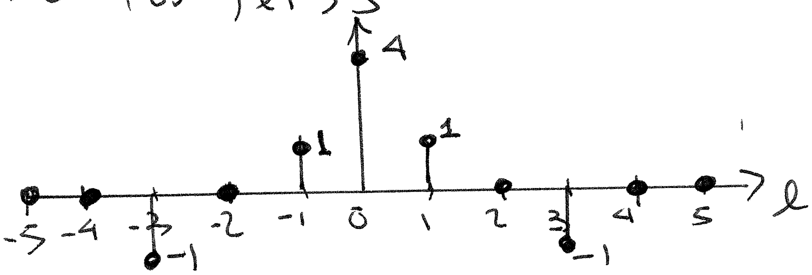
(ii) $r_{h_2h_2}[0] = 4$

$r_{h_2h_2}[1] = \begin{matrix}\{1, 1, 1, -1\} \\ \quad\{1, 1, 1, -1\}\end{matrix} = 1 = r_{h_2h_2}[-1]$

$r_{h_2h_2}[2] = \begin{matrix}\{1, 1, 1, -1\} \\ \qquad 1, 1, 1, -1\end{matrix} = 0 = r_{h_2h_2}[-2]$

$r_{h_2h_2}[3] = \begin{matrix}\{1, 1, 1, -1\} \\ \qquad\quad 1, 1, 1, -1\end{matrix} = -1 = r_{h_2h_2}[-3]$

$r_{h_2h_2}[\ell] = 0$ for $|\ell| > 3$



(iii) $r_{y_2y_2}[\ell] = r_{xx}[\ell] * r_{h_2h_2}[\ell] = \delta[\ell] * r_{h_2h_2}[\ell] = r_{h_2h_2}[\ell]$

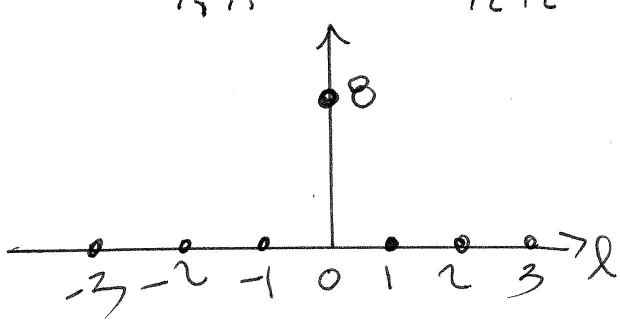
stem plot same as above

(c) $r_{y_1y_1}[\ell] + r_{y_2y_2}[\ell] = r_{h_1h_1}[\ell] + r_{h_2h_2}[\ell] = 8\delta[\ell]$

$= \{1, 0, -1, 4, -1, 0, 1\}$

$+ \{-1, 0, 1, 4, 1, 0, -1\}$

$= \{0, 0, 0, \underset{\substack{\uparrow \\ \ell=0}}{8}, 0, 0, 0\}$



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Problem 2. [20 points] System 1 and System 2 are as defined in parts (b) and (c) of Problem 1, respectively, but now they are connected in parallel, such that the overall output is

$$y[n] = y_1[n] + y_2[n]$$

(a) Determine the overall response $h[n]$ of the system. Do a stem plot of $h[n]$.

(b) Plot the magnitude of the frequency response of the system $|H(\omega)|$ over $-\pi < \omega < \pi$.

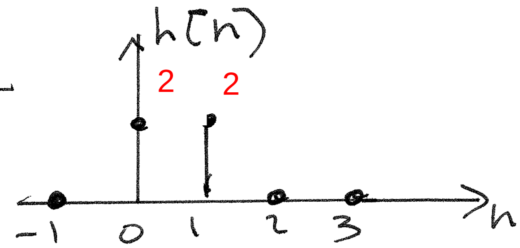
(c) Determine a closed-form expression for the output $y[n]$ when the input is $x[n]$ below.

$$x[n] = 1 + (j)^n + (-j)^n + (-1)^n$$

(a) $h_o[n] = h_1[n] + h_2[n] \Rightarrow$ parallel

$$= \{1, 1, -1, 1\}$$
$$+ \{1, 1, 1, -1\}$$
$$= \{\underset{\uparrow}{2}, 2\}$$

$2\,\delta[n] + \delta[n-1]\,2$



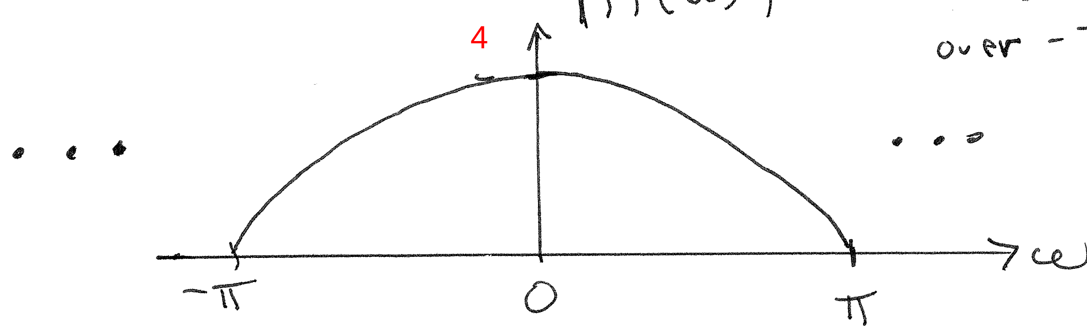
(b) $H(\omega) = 2 + 2e^{-j\omega}$

$$= 4\, e^{-j\frac{\omega}{2}} \left\{ \frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2} \right\}$$

$$= 4 \cos\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}}$$

$$|H(\omega)| = 4 \cos\left(\frac{\omega}{2}\right)$$

over $-\pi < \omega < \pi$



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Since $e^{j\frac{\pi}{2}n} = j^n$ and $(-1)^n = e^{j\pi n}$

(c) $y(n) = H(0)\cdot 1 + H(\frac{\pi}{2}) j^n$

$+ H(-\frac{\pi}{2})(-j)^n + \underbrace{H(\pi)}_{=0}(-1)^n$

$\Rightarrow H(0)=4 \qquad H(\pi)=0$

$H(\frac{\pi}{2}) = 4\cos(\frac{\pi}{4}) e^{-j\frac{\pi}{4}} = 1 + e^{-j\frac{\pi}{2}} = 2 - j2$

easier to go back to original expression

$H(\omega) = 1 + e^{-j\omega}$

$H(-\frac{\pi}{2}) = 2 + j2$

$H(-\omega) = H^*(\omega)$ for real-valued $h(n)$

$y(n) = 4 + 2(1-j)j^n + 2(1+j)(-j)^n$

NAME: KEY

Problem 3. [40 points]

(a) The autocorrelation of $x[n]$ may be computed in terms of convolution as

$$r_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

Which of the three main properties of convolution discussed in class is key to proving that the autocorrelation of $y[n] = x^*[-n]$ is the same as the autocorrelation of $x[n]$. Write your one-word answer directly below.

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$$r_{yy}[\ell] = y[\ell] * y^*[-\ell] = x^*[-\ell] * x[\ell] = x[\ell] * x^*[-\ell] = r_{xx}[\ell]$$

(b) Is the convolution of two all-pass filters/sequences also an all-pass filter/sequence? Prove your answer with a short mathematical analysis in the space provided on the next few blank pages.

(c) Let $x[n]$ be the input signal described below with $p = \frac{1}{2}$.

$$x[n] = \frac{1}{p}\left\{\delta[n] + (p^2 - 1)p^n u[n]\right\} \tag{2}$$

This signal is input to the system below described the difference equation below

$$y[n] = \frac{2}{3}y[n-1] - \frac{2}{3}x[n] + x[n-1]$$

(i) Draw the pole-zero diagram for this system. Is the system stable?

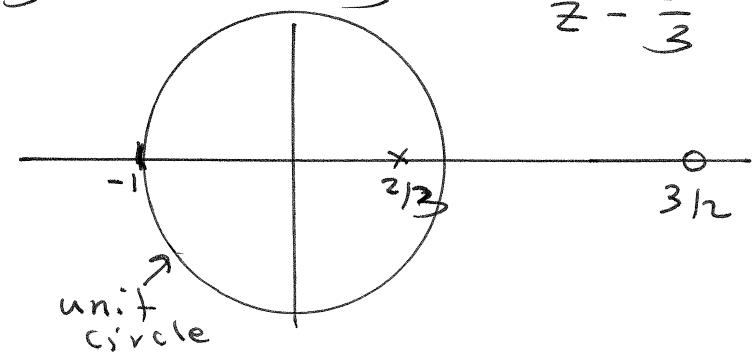
(ii) Determine the autocorrelation of the output $y[n]$. Do a stem plot of $r_{yy}[\ell]$.

(d) For the same input signal and same system described in part (c), determine the cross-correlation $r_{yx}[\ell]$ between the input and output. Just provide a closed-form expression for $r_{yx}[\ell]$; you don't need to plot it.

(c) $Y(z) = \frac{2}{3} z^{-1} Y(z) - \frac{2}{3} X(z) + z^{-1} X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{2}{3} + z^{-1}}{1 - \frac{2}{3}z^{-1}} = \frac{-\frac{2}{3}z + 1}{z - \frac{2}{3}} = \frac{-\frac{2}{3}\left(z - \frac{3}{2}\right)}{z - \frac{2}{3}}$$

zero at $3/2$
pole at $2/3$ } all-pass filter!



(b) Consider $x[n]$ to be "all-pass signal"
such that $|X(\omega)| = 1$ for all ω ($\forall \omega$)
Consider $h[n]$ to be all-pass filter
such that $|H(\omega)| = 1 \ \forall \omega$

Now, let
$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(\omega) = X(\omega) H(\omega)$$
Since $|ab| = |a||b|$, it follows that:
$$|Y(\omega)| = |H(\omega)|\,|X(\omega)| = 1 \cdot 1 = 1 \ \forall \omega$$

So $y[n]$ is all-pass with $r_{yy}[\ell] = \delta[\ell]$

Back to part (ii) of part (c)
since $x[n] = \frac{1}{p}\left\{\delta[n] + (p^2-1)\, p^n u[n]\right\}$ is all-pass
and $h[n]$ is all-pass (check value of $H(\omega)\big|_{\omega=0}$
$$= H(z)\Big|_{z=e^{j0}=1} = \frac{-\frac{2}{3}+1}{1-\frac{2}{3}} = 1 \ )$$
Thus, $y[n] = h[n] * x[n]$ is all-pass with $r_{yy}[\ell] = 1$

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(d) $r_{yx}[\ell] = h[\ell] * r_{xx}[\ell]$

$$= h[\ell] * \delta[\ell]$$

$$= h[\ell] = -\frac{1}{p}\left\{\delta[\ell] + (p^2-1)\,p^{\ell}\,u[\ell]\right\}$$

with $p = 2/3$

Check:

$$H(z) = -\frac{1}{p} + \frac{(1-p^2)}{p}\,\frac{z}{z-p} = \frac{-\frac{1}{p}(z-p) + \left(\frac{1}{p}-p\right)z}{z-p}$$

$$= \left.\frac{-pz+1}{z-p}\right|_{p=2/3} = \frac{-\frac{2}{3}z+1}{z-\frac{2}{3}}$$ checks!