Solution to Exam 1 Fall 2009

Sol'n to Prob. 1

We recognize $H(z) = \dfrac{z - \frac{1}{a}}{z - a}$ as the transfer function of an all-pass filter,

Thus, $|H(\omega)| =$ constant for all ω

Parts (a) and (b) are about proving this "graphically"

(a) $N(\omega) = \left| e^{j\omega} - \frac{1}{a} \right|$ a is real-valued

$$= \sqrt{\left(\cos\omega - \tfrac{1}{a}\right)^2 + \sin^2\omega}$$

$$= \sqrt{1 - 2\tfrac{1}{a}\cos\omega + 1/a^2}$$ since $\sin^2\theta + \cos^2\theta = 1$

(b) $D(\omega) = \left| e^{j\omega} - a \right|$

$$= \sqrt{(\cos\omega - a)^2 + \sin^2\omega}$$

$$= \sqrt{1 - 2a\cos\omega + a^2}$$

(c) factor out $\frac{1}{a^2}$ inside square root

$$N(\omega) = \frac{1}{a}\sqrt{a^2 - 2a\cos\omega + 1}$$

since $0 < a < 1$ THUS: $\dfrac{N(\omega)}{D(\omega)} = \dfrac{1}{a}$

Soln to Prob. 1 (cont.)

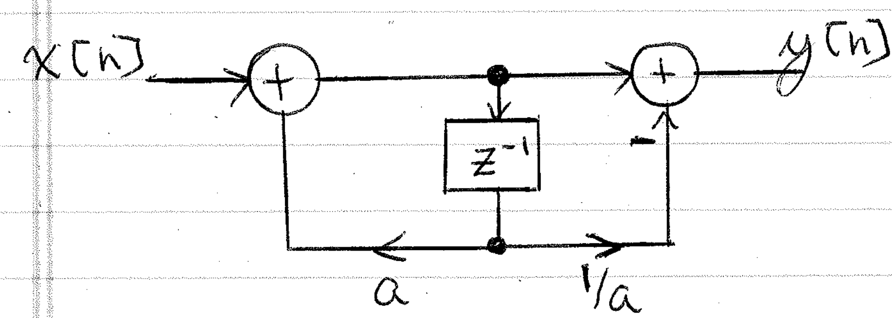
(d) $H(z) = \dfrac{z}{z-a} - \dfrac{1}{a} z^{-1} \dfrac{z}{z-a}$

$$h[n] = a^n u[n] - \frac{1}{a} a^{n-1} u[n-1]$$

(e) Since $|H(\omega)| = \dfrac{1}{a} \Rightarrow |H(\omega)|^2 = \dfrac{1}{a^2}$
$\forall \omega$

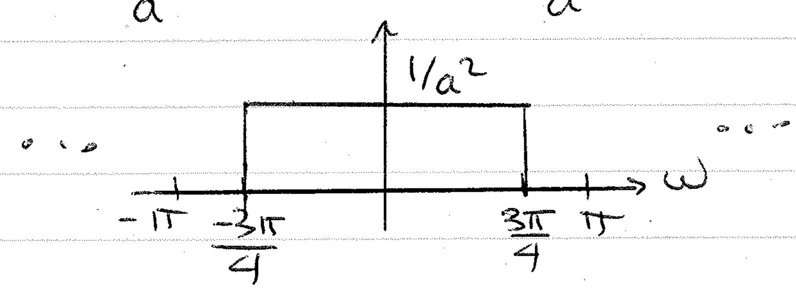
Thus, $r_{hh}[\ell] = \dfrac{1}{a^2} \delta[\ell]$ $\quad$ = constant

(f) Direct Form II from book:



(g) $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$

$$= \frac{1}{a^2} S_{xx}(\omega) = \frac{1}{a^2} |X(\omega)|^2$$



Parseval:

$$E_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(\omega)|^2 d\omega = \frac{1}{2\pi} \left(\frac{6\pi}{4}\right)\left(\frac{1}{a^2}\right) = \frac{3}{4} \frac{1}{a^2}$$

Solution to Exam 1

Sol'n to Prob. 2

(a) For real-valued sequences:

$$r_{xy}[l] = x[l] * y[-l]$$

$$r_{yx}[l] = y[l] * x[-l]$$

If both $x[n] = x[-n]$ and $y[n] = y[-n]$ are even-symmetric

$$r_{xy}[l] = x[l] * y[l]$$

$$r_{yx}[l] = y[l] * x[l]$$

$$= x[l] * y[l]$$ since convolution is commutative

Thus, $r_{xy}[l] = r_{yx}[l]$

(b) $r_{zz}[l] = z[l] * z^*[-l]$

$$= \left(x[l] + j\,y[l]\right) * \left(x[-l] - j\,y[-l]\right)$$

$$= x[l] * x[-l] + y[l] * y[-l]$$

$$+ j\left(y[l] * x[-l] - x[l] * y[-l]\right)$$

$$= r_{xx}[l] + r_{yy}[l] + j\left(r_{yx}[l] - r_{xy}[l]\right)$$

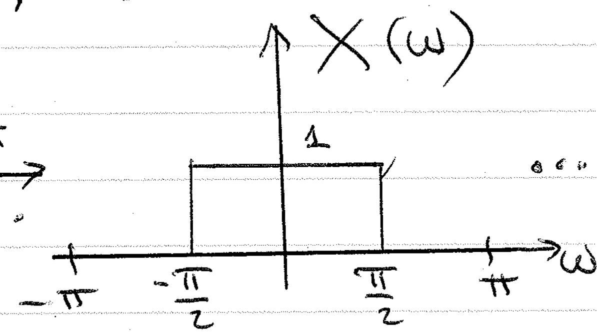
Sol'n to Prob. (2) (cont.)

Note, if $x[-n] = x[n]$ and $y[n] = y[-n]$, then it follows from part (a):

$$r_{zz}[\ell] = r_{xx}[\ell] + r_{yy}[\ell]$$

(c)

$$x[n] = \frac{\sin\left(\frac{\pi}{2} n\right)}{\pi n} \xleftrightarrow{DTFT} \ldots$$



Since the "height" is 1, $|X(\omega)|^2 = X(\omega)$

Thus, since $r_{xx}[\ell] \xleftrightarrow{DTFT} |X(\omega)|^2$

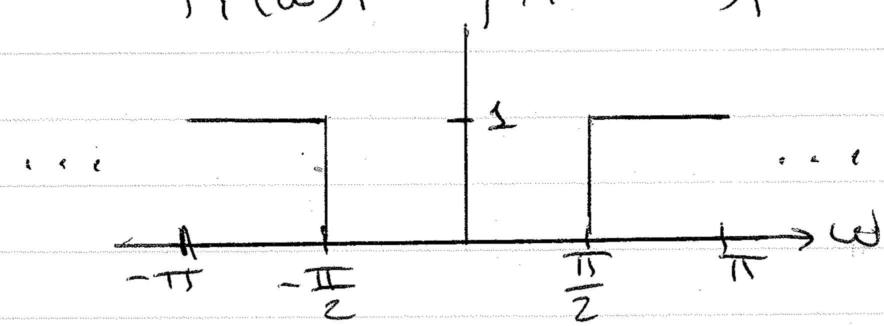
it follows that:

$$r_{xx}[\ell] = x[\ell] = \frac{\sin\left(\frac{\pi}{2}\ell\right)}{\pi \ell}$$

(d) $y[n] = (-1)^n x[n] = e^{j\pi n} x[n]$

Thus, $Y(\omega) = X(\omega - \pi)$

$$|Y(\omega)|^2 = |X(\omega - \pi)|^2$$



Sol'n to Prob. (2) (cont.)

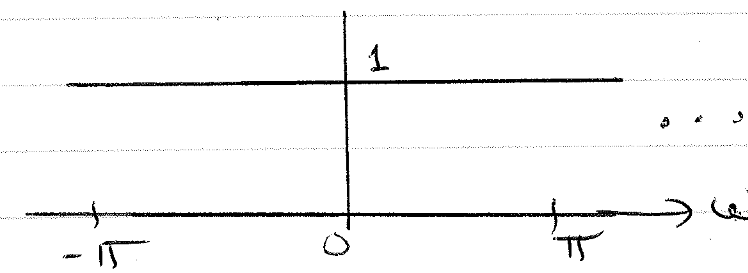
(d) Thus, $r_{yy}[\ell] = y[\ell]$

$$= (-1)^{\ell} \frac{\sin\left[\frac{\pi}{2}\ell\right]}{\pi \ell}$$

(e) From parts (a) and (b), since $x[n]$ in part (c) AND $y[n]$ in part (d) are both symmetric:

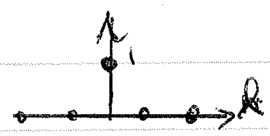
$$r_{zz}[\ell] = r_{xx}[\ell] + r_{yy}[\ell]$$

$$\xleftrightarrow{\text{DTFT}} |X(\omega)|^2 + |Y(\omega)|^2 = S_{zz}(\omega)$$



Thus,

$$r_{zz}[\ell] = \delta[\ell]$$



We can also see this by noting $\frac{\sin\left(\frac{\pi}{2}\ell\right)}{\pi \ell}$ is zero for even lag values AND

$$r_{xx}[\ell] + r_{yy}[\ell] = \underbrace{\left\{1 + (-1)^{\ell}\right\}}_{\substack{= 0 \text{ for odd} \\ \text{values of } \ell}} \frac{\sin\left(\frac{\pi}{2}\ell\right)}{\pi \ell}$$

Thus,

$$r_{xx}[\ell] + r_{yy}[\ell] = \delta[\ell]$$