Solution to Exam 1  Fall 2009
Sol'n to Prob. 1
We recognize \( H(z) = \frac{z - \frac{1}{a}}{z - a} \) as the transfer function of an all-pass filter.
Thus, \( |H(\omega)| = \text{constant for all } \omega \)
Parts (a) and (b) are about proving this "graphically."

(a) \( N(\omega) = |e^{j\omega} - \frac{1}{a}| \) \( a \) is real-valued

\[
= \sqrt{(\cos \omega - \frac{1}{a})^2 + \sin^2 \omega}
\]
\[
= \sqrt{1 - 2\frac{1}{a} \cos \omega + \frac{1}{a^2}}
\]
since \( \sin^2 \theta + \cos^2 \theta = 1 \)

(b) \( D(\omega) = |e^{j\omega} - a| \)

\[
= \sqrt{(\cos \omega - a)^2 + \sin^2 \omega}
\]
\[
= \sqrt{1 - 2a \cos \omega + a^2}
\]

(c) factor out \( \frac{1}{a^2} \) inside square root

\[
N(\omega) = \frac{1}{a} \sqrt{a^2 - 2a \cos \omega + 1}
\]
since \( 0 < a < 1 \)

Thus: \( \frac{N(\omega)}{D(\omega)} = \frac{1}{a} \)
(d) $H(z) = \frac{z}{z-a} - \frac{1}{a} \frac{z^{-1}}{z-a}$ 

$h[n] = a^n u[n] - \frac{1}{a} a^{n-1} u[n-1]$

(e) Since $|H(\omega)| = \frac{1}{a} \implies |H(\omega)|^2 = \frac{1}{a^2}$

Thus, $r_{hh}[\ell] = \frac{1}{a^2} \delta[\ell] = \text{constant}$

(f) Direct Form II from book:

![Direct Form II Diagram]

(g) $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \frac{1}{a^2} S_{xx}(\omega) = \frac{1}{a^2} |X(\omega)|^2$

Pause:

$E_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(\omega)|^2 d\omega = \frac{1}{2\pi} \left( \frac{6\pi}{a^2} \right) \left( \frac{1}{a^2} \right) = \frac{3}{4} \frac{1}{a^2}$
Solution to Exam 1

Soln to Prob. 2

(a) For real-valued sequences:

\[ r_{xy}[l] = x[l] * y[-l] \]

\[ r_{yx}[l] = y[l] * x[-l] \]

If both \( x[n] = x[-n] \) and \( y[n] = y[-n] \)
are even symmetric

\[ r_{xy}[l] = x[l] * y[l] \]

\[ r_{yx}[l] = y[l] * x[l] \]

= \( x[l] * y[l] \) since convolution is commutative

Thus, \( r_{xy}[l] = r_{yx}[l] \)

(b) \( r_{zz}[l] = z[l] * z^*[\text{z} - l] \)

= \((x[l] + j y[l]) * (x[-l] - j y[-l])\)

= \( x[l] * x[-l] + y[l] * y[-l]\)

+ \( j (y[l] * x[-l] - x[l] * y[-l])\)

= \( r_{xx}[l] + r_{yy}[l] + j (r_{yx}[l] - r_{xy}[l]) \)
S0l'n to Prob. (2) (cont.)

Note: if \( x(-n) = x(n) \) and \( y(n) = y(-n) \),
then it follows from part (a):

\[
R_{zz}[l] = R_{xx}[l] + R_{yy}[l]
\]

(c) \[
x[n] = \sin\left(\frac{\pi n}{2}\right) \quad \text{DTFT} \quad \frac{1}{\pi n} \]

Since the "height" is 1, \( |X(\omega)|^2 = X(\omega) \)

Thus, since \( R_{xx}[l] \leftrightarrow |X(\omega)|^2 \)

it follows that:

\[
R_{xx}[l] = \alpha[l] = \frac{\sin\left(\frac{\pi l}{2}\right)}{\pi l}
\]

(d) \( y[n] = (-1)^n x[n] = e^{j\pi n} x[n] \)

Thus, \( Y(\omega) = X(\omega - \pi) \)

\[
|Y(\omega)|^2 = |X(\omega - \pi)|^2
\]
(d) Thus, $R_{yy}[l] = \gamma[l]$

$$= (-1)^l \frac{\sin \left( \frac{\pi}{2} l \right)}{\pi l}$$

(e) From parts (a) and (b), since $x[n]$ in part (c) AND $y[n]$ in part (d) are both symmetric:

$$R_{zz}[l] = R_{xx}[l] + R_{yy}[l]$$

DTFT

$$\left| X(w) \right|^2 + \left| Y(w) \right|^2 = \sum_{r=\pm} R_{zz}(w)$$

Thus,

$$R_{zz}[l] = \delta[l]$$

We can also see this by noting $\frac{\sin \left( \frac{\pi}{2} l \right)}{\pi l}$ is zero for even large values AND

$$R_{xx}[l] + R_{yy}[l] = \left\{ 1 + (-1)^l \right\} \frac{\sin \left( \frac{\pi}{2} l \right)}{\pi l}$$

$$= 0 \text{ for odd values of } l$$

Thus,

$$R_{xx}[l] + R_{yy}[l] = \delta[l]$$