Exam 1 Fall 2006 Solution

Prob. 1 (a)

$$r_{xx}[m] = u[m+2] - u[m-3]$$

$$= \{1, 1, \underset{\uparrow}{1}, 1, 1\} \xleftrightarrow{\text{DTFT}} S_{xx}(\omega) = \underbrace{\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}}_{<0 \text{ for certain regions of } \omega}$$

BUT:

$$S_{xx}(\omega) = |X(\omega)|^2$$

> 0 for all ω

$\Rightarrow$ <u>not</u> valid

Prob. 1 (b)

$$r_{xx}[m] = (3-|m|)\left(u[m+2]-u[m-3]\right)$$

$$= \{1, \underset{\uparrow}{1}, 1\} * \{1, \underset{\uparrow}{1}, 1\}$$

$$\overset{DTFT}{\longleftrightarrow} \left(\frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right)^2 > 0 \text{ for all } \omega$$

$\Rightarrow$ valid autocorrelation sequence

Prob. 1 (c) $y[n] = x[n-n_0]$

$$Y(\omega) = X(\omega) e^{-j\omega n_0}$$

$$S_{yy}(\omega) = |Y(\omega)|^2 = |X(\omega)|^2$$

$$\Rightarrow r_{yy}[m] = r_{xx}[m]$$

$\Rightarrow$ different sequences can have same autocorrelation function

Prob. 1 (e) $\quad y[n] = e^{j(\omega_0 n + \theta)} x[n]$ 4

$$Y(\omega) = e^{j\theta} X(\omega - \omega_0)$$

$$S_{yy}(\omega) = |Y(\omega)|^2 = |X(\omega - \omega_0)|^2$$

taking inverse DTFT:

$$r_{yy}[m] = e^{j\omega_0 m} r_{xx}[m]$$

Prob. 1 (e)

Note: $X(z) = \frac{z}{z-a} - \frac{1}{a} z^{-1} \frac{z}{z-a}$

$= \frac{z - \frac{1}{a}}{z-a}$ } "all-pass"

$\Rightarrow |X(\omega)|^2 = \text{constant} \ \forall \omega$

at $\omega = 0 \Rightarrow \frac{e^{j0} - 1/\tfrac{1}{2}}{e^{j0} - \frac{1}{2}} = -2$

$|X(\omega)|^2 = 4 \ \forall \omega$

$\Rightarrow$ taking inverse DTFT $\Rightarrow$

(c)

$r_{xx}[m]$

$= 4\delta[m]$

Prob. 1 (e) (ii)

$$r_{yx}[m] = h[m] * r_{xx}[m]$$

$$= h[m] * 4\delta[m]$$

$$= 4\, h[m]$$

· impulse response: $H(z) = \dfrac{1}{1-\frac{1}{4}z^{-1}}$

$$h[n] = \left(\tfrac{1}{4}\right)^n u[n]$$

$$r_{yx}[m] = 4\left(\tfrac{1}{4}\right)^m u[m]$$

Prob. 1 (e) (iii)

$$r_{yy}[m] = r_{hh}[m] * r_{xx}[m]$$

$$= \frac{1}{1-\left(\frac{1}{4}\right)^2}\left(\frac{1}{4}\right)^{|m|} * 4\,\delta[m]$$

$$= \frac{64}{15}\left(\frac{1}{4}\right)^{|m|}$$

Prob. 2 $x_r(t) = s(t) * g(t)$ 8

$$= \left\{ \sum_k x[k]\, \delta(t - kT_s) \right\} * p(t) * g(t)$$

$$= \sum_k x[k]\, h(t - kT_s)$$

where: $h(t) = p(t) * g(t)$

$$= p(t) + p(t - T_s)$$

(9)

$$x_r[n] = x_r\left(n\frac{T_s}{L}\right)$$

$$= \sum x[k]\, h\left(n\frac{T_s}{L} - kT_s\right)$$

$$= \sum x[k]\, h[n-kL]$$

where: $h[n] = h\left(n\frac{T_s}{L}\right)$

$$= p\left(n\frac{T_s}{L}\right) + p\left(n\frac{T_s}{L} - T_s\right)$$

$$h[n] = p[n] + p[n-L]$$

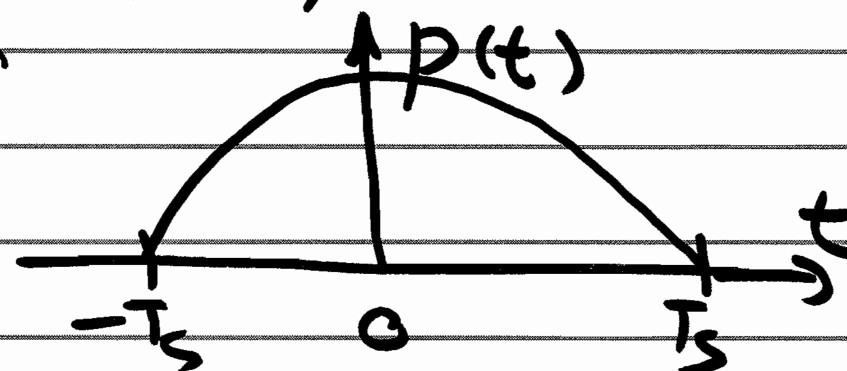
where: $p[n] = p\left(n\frac{T_s}{L}\right)$

(a) $L = 1$:

$$x_r[n] = p[n] + p[n-1]$$

$p[n] \Rightarrow$ sample $p(t)$ every T_s secs.

$$p[n] = \cos\left(\frac{\pi n T_s}{2 T_s}\right)$$



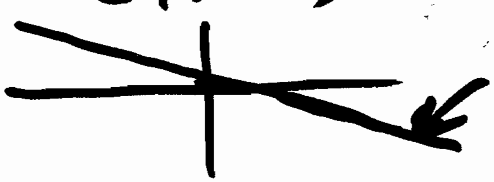
$$p[n] = \delta[n]$$

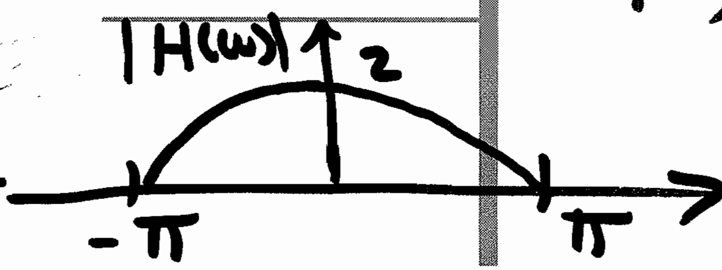
$\boxed{\text{only 1 nonzero sample}}$

$$h[n] = \delta[n] + \delta[n-1]$$

$$H(\omega) = 1 + e^{-j\omega} = 2\cos(\omega/2)\, e^{-j\frac{\omega}{2}}$$

$\angle H(\omega)$



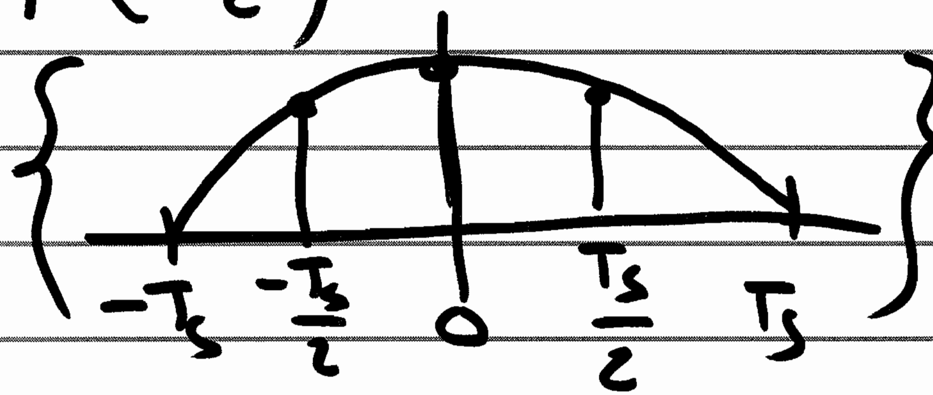


Prob. 2 (b)

$$h[n] = p[n] + p[n-2]$$

where $p[n] = p\left(n\frac{T_s}{2}\right)$

$$= \cos\left(\frac{\pi n \, T_s/2}{2T_s}\right)$$



$\Rightarrow$ 3 nonzero points

$$= \cos\left(\frac{\pi}{4}n\right)\left(u[n+1] - u[n-2]\right)$$

$\Rightarrow$ only nonzero for $-1 \le n \le 1$

Prob. 2 (b) (cont.)

$$H(\omega) = P(\omega)\left(1 + e^{-j2\omega}\right)$$

where: $$P(\omega) = \frac{1}{2}\,\frac{\sin\left(\frac{3}{2}\left(\omega - \frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{\pi}{4}\right)\right)} + \frac{1}{2}\,\frac{\sin\left(\frac{3}{2}\left(\omega + \frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega + \frac{\pi}{4}\right)\right)}$$

$$H(\omega) = P(\omega)\, 2\cos(\omega)\, e^{-j\omega}$$

let $p[n] = \left\{ \frac{1}{\sqrt{2}}, \underset{\uparrow}{1}, \frac{1}{\sqrt{2}} \right\}$

$$h[n] = p[n] + p[n-2]$$

$$= \left\{ \frac{1}{\sqrt{2}}, \underset{\uparrow}{1}, \sqrt{2}, 1, \frac{1}{\sqrt{2}} \right\}$$

symmetric sequence about $n=1$

$$H(\omega) = \left\{ \sqrt{2} + 2\cos(\omega) + \sqrt{2}\cos(2\omega) \right\} e^{-j\omega}$$

$$|H(\omega)| = \left| \left(1 + \cos(2\omega)\right) + \sqrt{2}\cos(\omega) \right| \sqrt{2}$$

$$= \left| 2\cos^2(\omega) + \sqrt{2}\cos(\omega) \right| \sqrt{2}$$

$$= \sqrt{2} \left| \cos(\omega) \right| \left| 2\cos(\omega) + \sqrt{2} \right|$$

$$= 2 \left| \cos(\omega) \right| \left| 1 + \sqrt{2}\cos(\omega) \right|$$

$$h_0[n] = h[2n]$$

$$= \{\underset{\uparrow}{1}, 1\} = \delta[n] + \delta[n-1]$$

$$H_0(\omega) = 1 + e^{-j\omega} = 2\cos(\omega)e^{-j\frac{\omega}{2}}$$

see before

$$h_1[n] = h[1+2n]$$

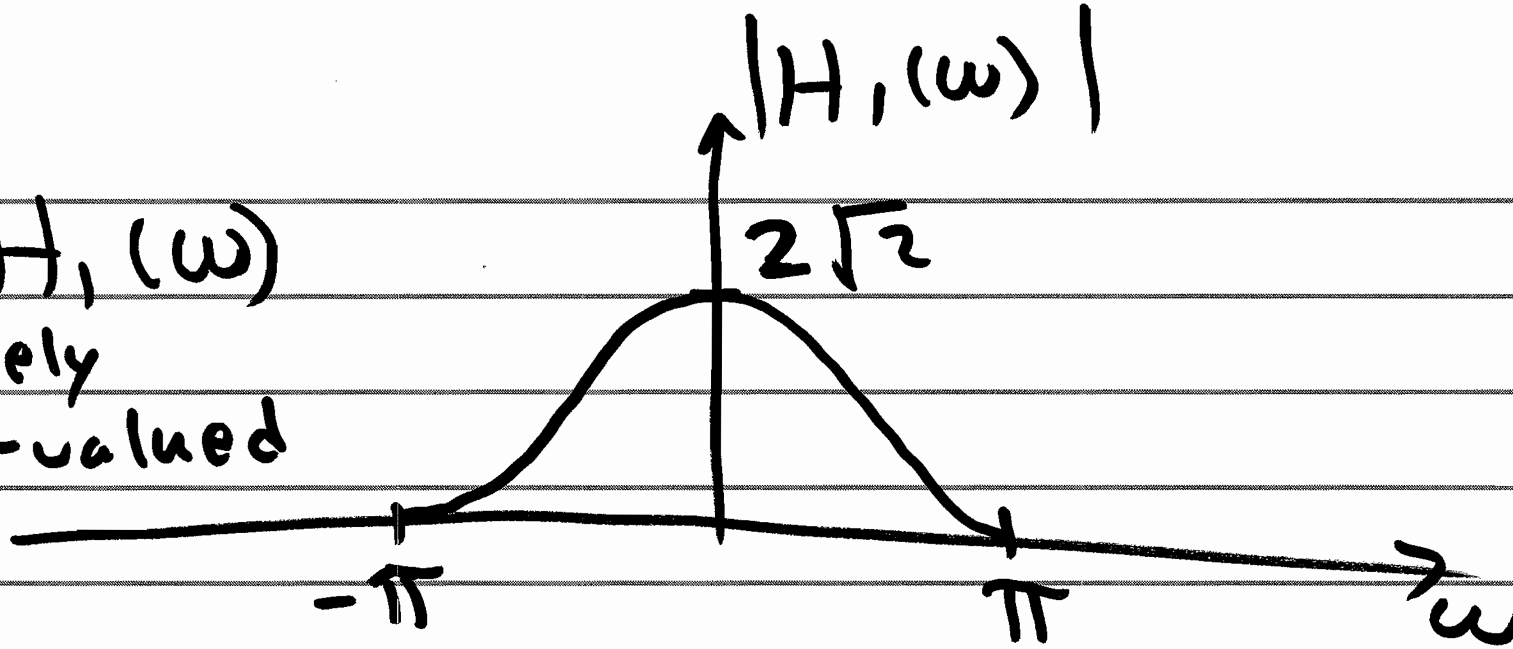
$$= \{\tfrac{1}{\sqrt{2}}, \underset{\uparrow}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\}$$

$$H_1(\omega) = \sqrt{2} + \sqrt{2}\cos(\omega)$$

$$= \sqrt{2}\,(1 + \cos(\omega))$$

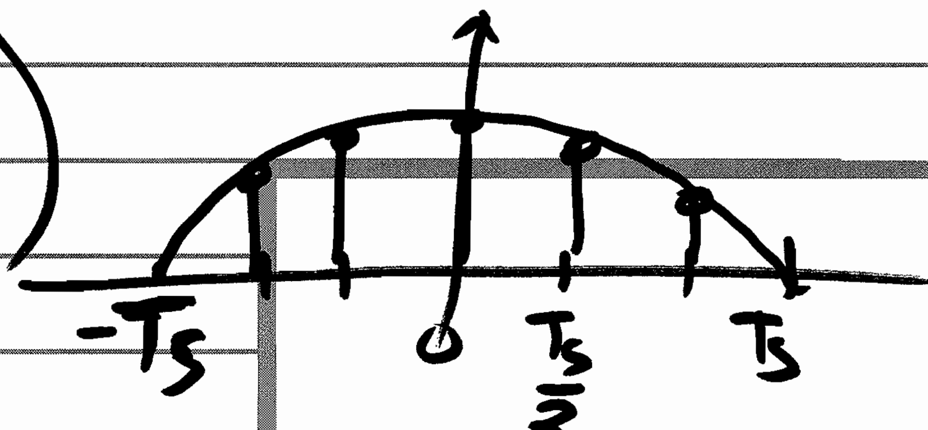
$H(\omega) = \frac{1}{\sqrt{2}}e^{j\omega} + \sqrt{2} + \frac{1}{\sqrt{2}}e^{-j\omega}$

$H_1(\omega)$
purely real-valued



(c) $L = 3$: $\quad h[n] = p[n] + p[n-3]$

$$p[n] = \cos\left(\frac{\pi n \frac{T_s}{3}}{2T_s}\right)$$



$$p[n] = \cos\left(\frac{\pi}{6} n\right) \cdot \left\{u[n+2] - u[n-3]\right\}$$

5 nonzero points

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$$h[n] = \cos\left(\frac{\pi}{6} n\right)\left(u[n+2] - u[n-3]\right)$$
$$+ \cos\left(\frac{\pi}{6}(n-3)\right)\left(u[n-1] - u[n-6]\right)$$

DONE