EE 538 — Homework # 2 Solutions — Fall 2000

Problem #1 : 3.43

$h(n) \sim$ LTI System

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$

$$y(n) = \left(\tfrac{1}{3}\right)^n u(n)$$

(a) Find $H(z)$

$$X(z) = \frac{z}{z-\frac{1}{2}} - \frac{1}{4} z^{-1} \cdot \frac{z}{z-\frac{1}{2}} = \frac{z-\frac{1}{4}}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$Y(z) = \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{z}{z-1/3}}{\frac{z-1/4}{z-1/2}} = \frac{z(z-\frac{1}{2})}{(z-\frac{1}{3})(z-\frac{1}{4})}, \quad |z| > \frac{1}{3}$$

Find $h(n)$:
$$\frac{H(z)}{z} = \frac{z(z-\frac{1}{2})}{z(z-\frac{1}{3})(z-\frac{1}{4})} = \frac{A}{z-\frac{1}{3}} + \frac{B}{z-\frac{1}{4}}$$

$$A = \frac{H(z)}{z} \cdot (z-\tfrac{1}{3})\Big|_{z=\frac{1}{3}} = \frac{\frac{1}{3}-\frac{1}{2}}{\frac{1}{3}-\frac{1}{4}} = -2$$

$$B = \frac{H(z)}{z} \cdot (z-\tfrac{1}{4})\Big|_{z=\frac{1}{4}} = \frac{\frac{1}{4}-\frac{1}{2}}{\frac{1}{4}-\frac{1}{3}} = 3$$

$$\therefore H(z) = \frac{-2z}{z-\frac{1}{3}} + \frac{3z}{z-\frac{1}{4}}$$

$$h(n) = \left(-2\left(\tfrac{1}{3}\right)^n + 3\left(\tfrac{1}{4}\right)^n\right) u(n)$$

(b) $$H(z) = \frac{Y(z)}{X(z)} = \frac{(z-\frac{1}{2})\cdot z}{(z-\frac{1}{3})(z-\frac{1}{4})}$$

$$Y(z)\left(z-\tfrac{1}{3}\right)\left(z-\tfrac{1}{4}\right) = X(z)\left(z-\tfrac{1}{2}\right)\cdot z$$

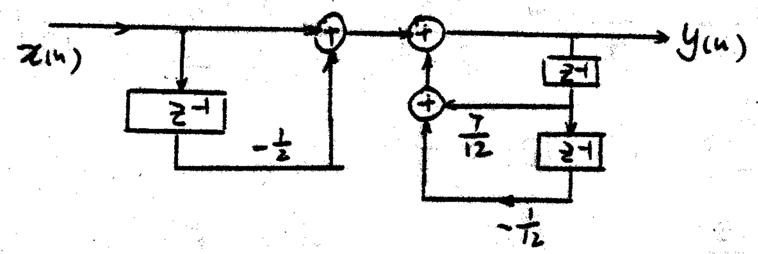
$$\frac{1}{z^2}\,Y(z)\left(z^2-\tfrac{7}{12}z+\tfrac{1}{12}\right) = X(z)\left(z^2-\tfrac{1}{2}z\right)\cdot\frac{1}{z^2}$$

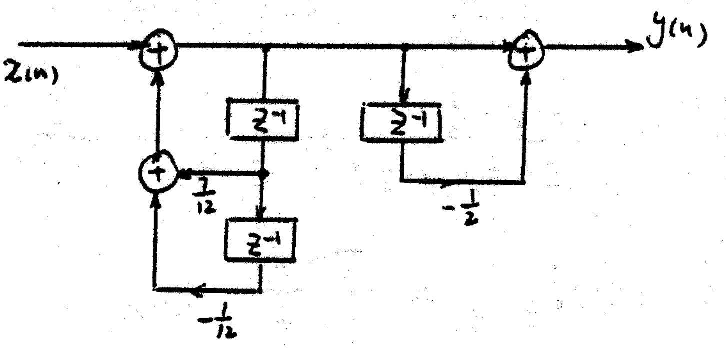
$$Y(z)\left(1-\tfrac{7}{12}z^{-1}+\tfrac{1}{12}z^{-2}\right) = X(z)\left(1-\tfrac{1}{2}z^{-1}\right)$$

$$\therefore\ y(n)-\tfrac{7}{12}y(n-1)+\tfrac{1}{12}y(n-2) = x(n)-\tfrac{1}{2}x(n-1)$$

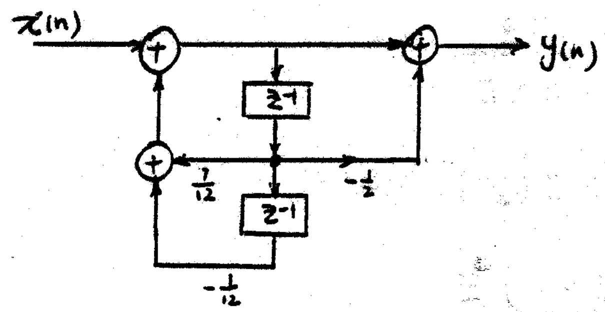
$$y(n) = x(n)-\tfrac{1}{2}x(n-1)+\tfrac{7}{12}y(n-1)-\tfrac{1}{12}y(n-2)$$

ⓒ Direct form I:





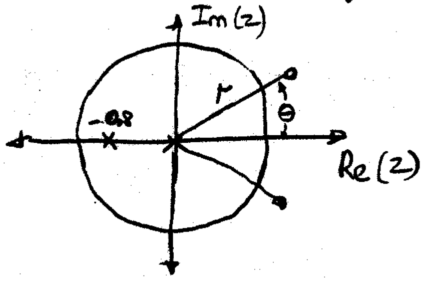
Direct form II



ⓓ Since the ROC of $H(z)$ contains the unit circle, the system is stable.

Problem # 2 3.49

A causal systen is defined by following pole-zero pattern.



a) $$H(z) = \frac{A(z - 1.5e^{j\pi/6})(z - 1.5e^{-j\pi/6})}{z(z+0.8)}$$ A- constant

$$H(1) = A\frac{\left(1 - \frac{3}{2}\cdot\frac{\sqrt{3}}{2} - j\frac{3}{2}\cdot\frac{1}{2}\right)\left(1 - \frac{3}{2}\cdot\frac{\sqrt{3}}{2} + j\frac{3}{2}\cdot\frac{1}{2}\right)}{1(1+0.8)} = 1$$

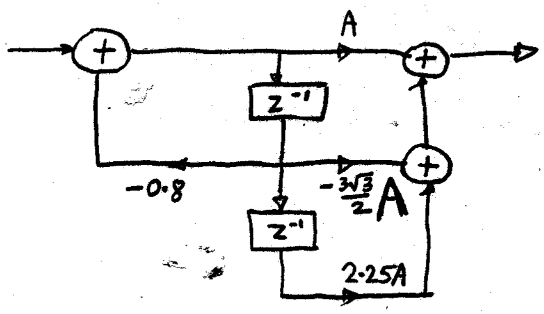
$$\Rightarrow A\frac{\left[\left(1 - \frac{3}{2}\cdot\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\cdot\frac{1}{2}\right)^2\right]}{1.8} = 1$$

$$A = \frac{1.8}{1 - \frac{3\sqrt{3}}{2} + 2.25}$$

$$H(z) = \frac{1.8}{1 - \frac{3\sqrt{3}}{2} + 2.25}\left[\frac{\left(z^2 - \frac{3\sqrt{3}}{2}z + 2.25\right)}{z(z+0.8)}\right].$$

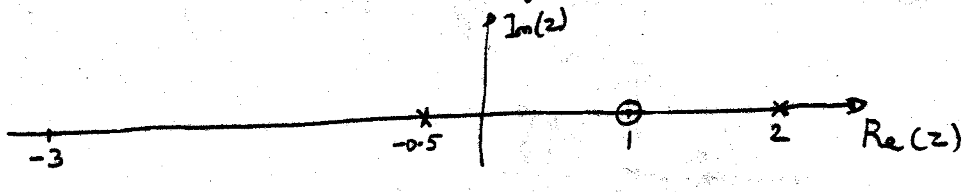
b) Since the system is causal and all poles of $H(z)$ are inside the unit circle, the system is stable.

c) $$y(n) = A\left[x(n) - \frac{3\sqrt{3}}{2}x(n-1) + 2.25\,x(n-2)\right] - 0.8\,y(n-1)$$



Problem # 3 (3.51)

A system has following pole-zero pattern

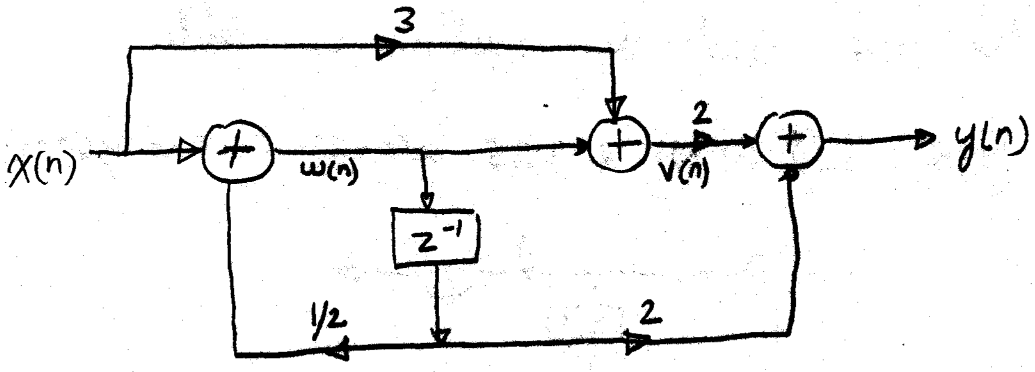


a) If the system is stable the ROC must be $0.5 < |z| < 2$ as it must contain the unit circle.

b) The only ROC possible for system to be causal is $|z| > 3$
As the ROC for a stable system is $0.5 < |z| < 2$
this system cannot correspond to a system which is both causal and stable

c) Four possible ROC's
$|z| > 3$, $2 < |z| < 3$, $\frac{1}{2} < |z| < 2$, $|z| < \frac{1}{2}$

Problem # 4 (7.3)



$$y(n) = 2v(n) + 2w(n-1) \overset{ZT}{\longleftrightarrow} Y(z) = 2V(z) + 2z^{-1}W(z)$$
$$v(n) = w(n) + 3x(n) \overset{ZT}{\longleftrightarrow} V(z) = W(z) + 3X(z)$$
$$w(n) = x(n) + \tfrac{1}{2}w(n-1) \overset{ZT}{\longleftrightarrow} W(z) = X(z) + \tfrac{1}{2}z^{-1}W(z)$$

$$W(z) = \frac{1}{1-\frac{1}{2}z^{-1}}X(z)$$ (Express $Y(z)$ in terms of $X(z)$)

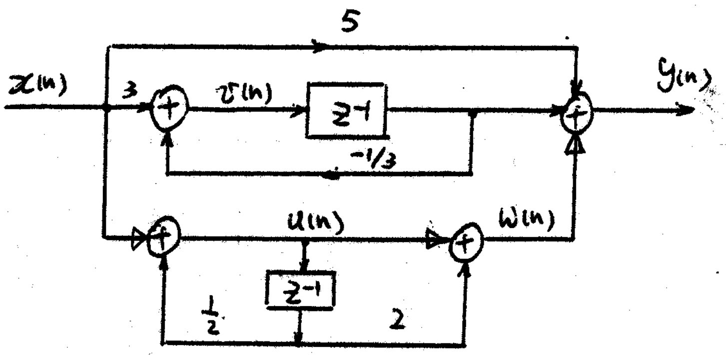
$$V(z) = \left[\frac{1}{1-\frac{1}{2}z^{-1}} + 3\right]X(z) = \frac{4-\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}X(z)$$

$$Y(z) = \left[2\cdot\frac{4-\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} + 2\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}\right]X(z) = \left[\frac{8-3z^{-1}+2z^{-1}}{1-\frac{1}{2}z^{-1}}\right]X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{8-z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{8z-1}{z-\frac{1}{2}}$$

$$h(n) = 8\cdot\left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{2}\right)^{n-1}u(n-1) \quad (\text{for ROC } |z| > \tfrac{1}{2})$$
$$h(n) = -8\cdot\left(\tfrac{1}{2}\right)^n u(-n-1) + \left(\tfrac{1}{2}\right)^{n-1}u(-n) \quad (\text{for ROC } |z| < \tfrac{1}{2})$$

Problem #5. (7.4)



$$y(n) = 5x(n) + v(n-1) + w(n) \overset{ZT}{\longleftrightarrow} Y(z) = 5X(z) + z^{-1}V(z) + W(z)$$
$$v(n) = 3x(n) - \tfrac{1}{3}v(n-1) \overset{ZT}{\longleftrightarrow} V(z) = 3X(z) - \tfrac{1}{3}z^{-1}V(z)$$
$$w(n) = u(n) + 2u(n-1) \overset{ZT}{\longleftrightarrow} W(z) = U(z) + 2z^{-1}U(z)$$
$$u(n) = x(n) + \tfrac{1}{2}u(n-1) \overset{ZT}{\longleftrightarrow} U(z) = X(z) + \tfrac{1}{2}z^{-1}U(z)$$

$$\therefore\ U(z)=\frac{1}{1-\frac{1}{2}z^{-1}}X(z)$$

$$W(z)=\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}X(z)$$

$$V(z)=\frac{3}{1+\frac{1}{3}z^{-1}}X(z)$$

$$Y(z)=5X(z)+\frac{3z^{-1}}{1+\frac{1}{3}z^{-1}}X(z)+\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}X(z)$$

$$\therefore\ H(z)=\frac{Y(z)}{X(z)}=5+\frac{3z^{-1}}{1+\frac{1}{3}z^{-1}}+\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$=5+\frac{3}{z+\frac{1}{3}}+\frac{z+2}{z-\frac{1}{2}}$$

$$h(n)=5\delta(n)+3\left(-\tfrac{1}{3}\right)^{n-1}u(n-1)+\left(\tfrac{1}{2}\right)^{n}u(n)+2\left(\tfrac{1}{2}\right)^{n-1}u(n-1).\quad (|z|>\tfrac{1}{2})$$

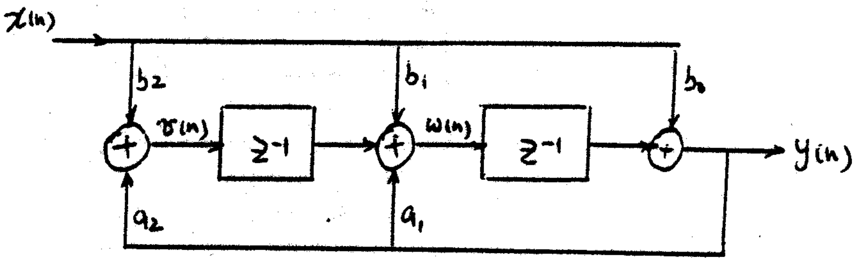
$$h(n)=5\delta(n)+3\left(-\tfrac{1}{3}\right)^{n-1}u(n-1)-\left(\tfrac{1}{2}\right)^{n}u(-n-1)-2\left(\tfrac{1}{2}\right)^{n-1}u(-n)$$

$$(\tfrac{1}{3}<|z|<\tfrac{1}{2})$$

$$h(n)=5\delta(n)-3\left(-\tfrac{1}{3}\right)^{n-1}u(-n)-\left(\tfrac{1}{2}\right)^{n}u(-n-1)-2\left(\tfrac{1}{2}\right)^{n-1}u(-n).\quad (|z|<\tfrac{1}{3})$$

Problem 6.

(7.7)



ⓐ $y(n)=b_0x(n)+w(n-1)$

$w(n)=b_1x(n)+v(n-1)+a_1y(n)$ —①

$v(n)=b_2x(n)+a_2y(n)$ —②

$\therefore\ y(n)=b_0x(n)+b_1x(n-1)+v(n-2)+a_1y(n-1)$ (From ①)

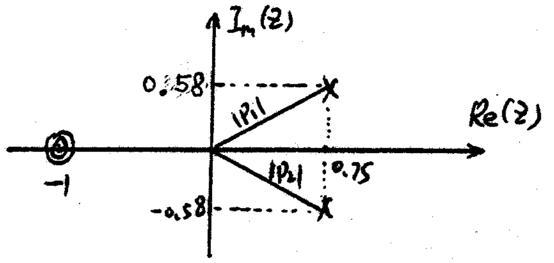
$=b_0x(n)+b_1x(n-1)+b_2x(n-2)+a_1y(n-1)+a_2y(n-2)$

(From ②)

$$\Rightarrow H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

(b) ① $b_0 = b_2 = 1, \quad b_1 = 2, \quad a_1 = 1.5 \quad a_2 = -0.9$

$$H(z) = \frac{z^2 + 2z + 1}{z^2 - 1.5z + 0.9} = \frac{(z+1)^2}{(z - 0.75 + j0.58)(z - 0.75 - j0.58)}$$

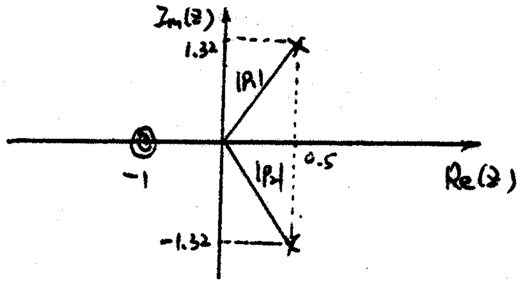


$|P_1| = |P_2| = \sqrt{0.9} < 1$

$\therefore$ stable system

② $b_0 = b_2 = 1, \quad b_1 = 2, \quad a_1 = 1, \quad a_2 = -2$

$$H(z) = \frac{z^2 + 2z + 1}{z^2 - z + 2} = \frac{(z+1)^2}{(z - 0.5 + j1.32)(z - 0.5 - j1.32)}$$



$|P_1| = |P_2| = \sqrt{2} > 1$

$\therefore$ unstable system

(c) $x(n) = \cos\left(\frac{\pi n}{3}\right) = \frac{1}{2}\left(e^{j\frac{\pi n}{3}} + e^{-j\frac{\pi n}{3}}\right)$

$$y(n) = \frac{1}{2} H(e^{j\frac{\pi}{3}})\, e^{j\frac{\pi}{3}n} + \frac{1}{2} H(e^{-j\frac{\pi}{3}})\, e^{-j\frac{\pi}{3}n}$$

$$H(z) = \frac{z^2}{z^2 - z + 0.99}, \quad H(e^{j\frac{\pi}{3}}) = \frac{e^{j\frac{2\pi}{3}}}{e^{j\frac{2\pi}{3}} - e^{j\frac{\pi}{3}} + 0.99} = \frac{-e^{-j\frac{\pi}{3}}}{-e^{-j\frac{\pi}{3}} - e^{j\frac{\pi}{3}} + 0.99}$$

$$= \frac{-e^{-j\pi/3}}{-2\cos\frac{\pi}{3} + 0.99} = \frac{-e^{-j\frac{\pi}{3}}}{-2\times 0.5 + 0.99} = 100\, e^{-j\frac{\pi}{3}}$$

$$H(e^{-j\frac{\pi}{3}}) = 100\, e^{j\frac{\pi}{3}}$$

$$\therefore y(n) = \frac{1}{2}\cdot 100\, e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{3}n} + \frac{1}{2}\cdot 100\, e^{j\frac{\pi}{3}} e^{-j\frac{\pi}{3}n}$$

$$y(n) = 100\cdot\frac{1}{2}\left(e^{j\frac{\pi}{3}(n-1)} + e^{-j\frac{\pi}{3}(n-1)}\right) = 100\cos\frac{\pi}{3}(n-1) \quad //$$