

## Sect. 5.3 Correlation Functions and Spectra at the output of LTI Systems

Addendum to  
Module 2 on  
Autocorrelation and Cross-Correlation

All of this is in  
Sect. 5.3 of the text book

Plus Energy Density Spectrum

Summarizing three main properties of auto correlation sequence  $r_{xx}[l] = x[l] * x^*[-l]$  (5)

1.  $r_{xx}[-l] = r_{xx}^*[l]$

2.  $|r_{xx}[l]| \leq r_{xx}[0]$

3.  $\sum_{l=-\infty}^{\infty} r_{xx}[l] e^{-j\omega l} \geq 0$

for all  $\omega$

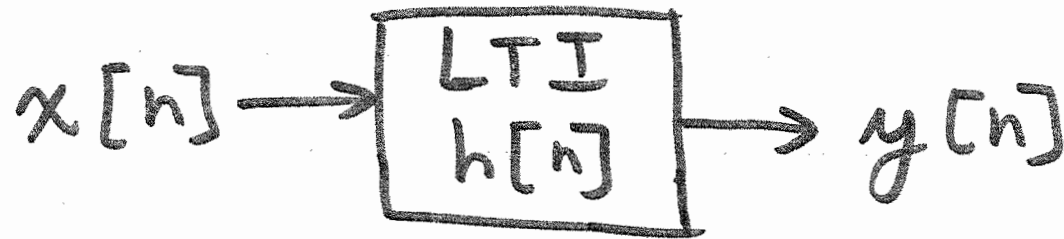
From 1. ,  $S_{xx}(\omega) = \sum_{l=-\infty}^{\infty} r_{xx}[l] e^{-j\omega l}$

is real-valued for all  $\omega$

$$S'_{xx}(\omega) = X(\omega) X^*(\omega) = |X(\omega)|^2$$

$\Rightarrow$  energy density spectrum

# I/O Relationships for LTI System



$$r_{yx}[\ell] = r_{xx}[\ell] * h[\ell] \quad \left. \vphantom{r_{yx}[\ell]} \right\} \begin{array}{l} \text{cross-} \\ \text{correlation} \\ \text{between input} \\ \text{and output} \end{array}$$

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

Energy Density Spectrum:

$$r_{xx}[\ell] \xleftrightarrow{\text{DTFT}} S_{xx}(\omega) = \mathcal{F}\{r_{xx}[\ell]\}$$

Props. (ii) + (iii):  $S_{xx}(\omega) = |X(\omega)|^2 \geq 0 \quad \forall \omega$

since  $x^*[-\ell] \xleftrightarrow{\text{DTFT}} X^*(\omega)$

# I/O Relationships for Energy Density Spectrum



$$r_{xx}[l] \xleftrightarrow{\text{DTFT}} \sum_{xx}(\omega) = |X(\omega)|^2$$

$$r_{yy}[l] \xleftrightarrow{\text{DTFT}} \sum_{yy}(\omega) = |Y(\omega)|^2$$

$$\sum_{yy}(\omega) = |H(\omega)|^2 \sum_{xx}(\omega)$$

Parseval's Theorem:

$$\text{Energy} = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$= r_{xx}[0] = E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{xx}(\omega) d\omega$$

## Frequency Domain Proofs of Key Properties of Autocorrelation

0. The DTFT of  $r_{xx}[l]$  is non-negative for all frequencies

$$r_{xx}[l] = x[l] * x^*[-l]$$

$$\xleftrightarrow{\text{DTFT}} \sum_{xx}(\omega) = X(\omega) X^*(\omega) = |X(\omega)|^2$$

Using the DTFT properties:

$$x^*[-n] \xleftrightarrow{\text{DTFT}} X^*(\omega)$$

$$\text{and } x[n] * y[n] \xleftrightarrow{\text{DTFT}} X(\omega) Y(\omega)$$

$\geq 0$   
for all  $\omega$

# Frequency Domain Proofs of Two Key Properties of Autocorrelation

$$1. \quad y[n] = x[n - n_0] \Rightarrow r_{yy}[l] = r_{xx}[l]$$

$$r_{yy}[l] = y[l] * y^*[-l]$$

$$\star \quad y[l] = x[l - n_0] \xleftrightarrow{\text{DTFT}} Y(\omega) = X(\omega) e^{-j\omega n_0}$$

$$\begin{aligned} S_{yy}(\omega) &= \widetilde{\mathcal{F}}\{r_{yy}[l]\} = X(\omega) e^{-j\omega n_0} \left\{ X^*(\omega) e^{+j\omega n_0} \right\} \\ &= |X(\omega)|^2 \end{aligned}$$

$$\text{Thus: } r_{yy}[l] = r_{xx}[l]$$

$$2. \quad y[n] = e^{j(\omega_0 n + \theta)} x[n]$$

$$\Rightarrow r_{yy}[l] = e^{j\omega_0 l} r_{xx}[l]$$

$$Y(\omega) = e^{j\theta} X(\omega - \omega_0)$$

$$\text{Thus: } |Y(\omega)|^2 = e^{j\theta} X(\omega - \omega_0) e^{-j\theta} X^*(\omega - \omega_0)$$

$$= |X(\omega - \omega_0)|^2$$

At the same time, modulation property of DTFT

$$e^{j\omega_0 l} r_{xx}[l] \xleftrightarrow{\text{DTFT}} |X(\omega - \omega_0)|^2$$

Thus:

$$r_{yy}[l] = e^{j\omega_0 l} r_{xx}[l]$$