Sect. 5.3 Correlation Functions and Spectra at the output of LTI Systems

Addendum to
Module 2 on
Autocorrelation and Cross-Correlation

All of this is in
Sect. 5.3 of the textbook

Plus Energy Density Spectrum

Summarizing three main properties of autocorrelation sequence $r_{xx}[\ell] = x[\ell] * x^*[-\ell]$

1. $r_{xx}[-\ell] = r_{xx}^*[\ell]$

2. $|r_{xx}[\ell]| \leq r_{xx}[0]$

3. $\sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell} \geq 0$

for all ω

From 1., $S_{xx}(\omega) = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell}$

is real-valued for all ω

$S_{xx}(\omega) = X(\omega) X^*(\omega) = |X(\omega)|^2$

$\Rightarrow$ energy density spectrum

I/O Relationships for LTI System

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$$

cross-correlation between input and output

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

Energy Density Spectrum:

$$r_{xx}[\ell] \xleftrightarrow{\text{DTFT}} S_{xx}(\omega) = \mathcal{F}\{r_{xx}[\ell]\}$$

Props. (ii) + (iii): $S_{xx}(\omega) = |X(\omega)|^2 \geq 0 \quad \forall \omega$

since $x^*[-\ell] \xleftrightarrow{\text{DTFT}} X^*(\omega)$

I/O Relationships for Energy Density Spectrum

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

$$r_{xx}[\ell] \xleftrightarrow{\text{DTFT}} S_{xx}(\omega) = |X(\omega)|^2$$

$$r_{yy}[\ell] \xleftrightarrow{\text{DTFT}} S_{yy}(\omega) = |Y(\omega)|^2$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

Parseval's Theorem:

$$\text{Energy} = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

$$= r_{xx}[0] = E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)\, d\omega$$

Frequency Domain Proofs of Key Properties of Autocorrelation

0. The DTFT of $r_{xx}[\ell]$ is non-negative for all frequencies

$$r_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

$$\overset{DTFT}{\longleftrightarrow} S_{xx}(\omega) = X(\omega)X^*(\omega) = \underbrace{|X(\omega)|^2}_{\substack{\geq 0 \\ \text{for all } \omega}}$$

Using the DTFT properties:

$$x^*[-n] \overset{DTFT}{\longleftrightarrow} X^*(\omega)$$

and $$x[n] * y[n] \overset{DTFT}{\longleftrightarrow} X(\omega)Y(\omega)$$

Frequency Domain Proofs of Two Key Properties of Autocorrelation

1. $y[n] = x[n-n_0] \Rightarrow r_{yy}[\ell] = r_{xx}[\ell]$

$$r_{yy}[\ell] = y[\ell] * y^*[-\ell]$$

$$\star \quad y[\ell] = x[\ell - n_0] \xleftrightarrow{DTFT} Y(\omega) = X(\omega)e^{-j\omega n_0}$$

$$S_{yy}(\omega) = \mathcal{F}\{r_{yy}[\ell]\} = X(\omega)e^{-j\omega n_0}\{X^*(\omega)e^{+j\omega n_0}\}$$

$$= |X(\omega)|^2$$

Thus: $r_{yy}[\ell] = r_{xx}[\ell]$

2. $y[n] = e^{j(\omega_0 n + \theta)} x[n]$

$$\Rightarrow r_{yy}[\ell] = e^{j\omega_0 \ell} r_{xx}[\ell]$$

$$Y(\omega) = e^{j\theta} X(\omega - \omega_0)$$

Thus: $|Y(\omega)|^2 = e^{j\theta} X(\omega - \omega_0) e^{-j\theta} X^*(\omega - \omega_0)$

$$= |X(\omega - \omega_0)|^2$$

At the same time, modulation property of DTFT

$$e^{j\omega_0 \ell} r_{xx}[\ell] \xleftrightarrow{DTFT} |X(\omega - \omega_0))^2$$

Thus:

$$r_{yy}[\ell] = e^{j\omega_0 \ell} r_{xx}[\ell]$$