• Key formulas for Multirate Analysis

$$x[n] \longrightarrow \boxed{L\uparrow} \longrightarrow y[n]$$

Insert L-1 zeros between each successive values of $x[n]$

Time Domain: $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

Frequency Domain: $$Y(\omega) = X(L\omega)$$

Decimator: $$y[n] = x[Dn]$$

$$x[n] \longrightarrow \boxed{D\downarrow} \longrightarrow$$

Frequency Domain: $$Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega - k2\pi}{D}\right)$$

- Related to the polyphase filters, we (2) encountered terms like:

$$h_\ell[n] = h[Ln+\ell] \xleftrightarrow{\text{DTFT}} H_\ell(\omega) = ?$$

- First consider: $g_\ell[n] = h[n+\ell]$

$$\Rightarrow \quad G_\ell(\omega) = e^{j\omega\ell} H(\omega)$$

- Next: $h_\ell[n] = g_\ell[Ln] = h[Ln+\ell]$

- Thus: $H_\ell(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} G_\ell\left(\frac{\omega - k2\pi}{L}\right)$

Substitute: $G_\ell(\omega) = e^{j\omega\ell} H(\omega)$

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$$H_\ell(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} e^{j\frac{(\omega - k2\pi)\ell}{L}} H\left(\frac{\omega - k2\pi}{L}\right)$$

$$= \left\{ \frac{1}{L} \sum_{k=0}^{L-1} e^{-j\frac{k2\pi\ell}{L}} H\left(\frac{\omega - k2\pi}{L}\right) \right\} e^{j\frac{\ell}{L}\omega}$$

In ideal case: $h[n] = \dfrac{\sin\left(\frac{\pi}{L}n\right)}{\frac{\pi}{L}n}$

in which case:

$$H_\ell(\omega) = e^{j\frac{\ell}{L}\omega} \quad \text{for } -\pi < \omega < \pi$$

corresponding to shift by fractional amount $(\ell/L)T_s$ back in the time domain

Alternative Derivation of DTFT of h[nL+ ell]

①

Note re: efficient polyphase implementation of up-sampling by a factor of L:

$$h[nL+\ell] \xleftrightarrow{\text{DTFT}} H_\ell(\omega) = ?$$

Recall:

$$\sum_{k=0}^{L-1} \frac{1}{L} e^{j\frac{2\pi}{L}kn} = \sum_{k=-\infty}^{\infty} \delta[n-kL]$$

Thus:

$$\frac{1}{L}\sum_{k=0}^{L-1} e^{j\frac{2\pi k}{L}(n-\ell)} = \begin{cases} 1, & \text{when } n = mL+\ell \\ 0, & \text{otherwise} \end{cases}$$

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$$H_\ell(\omega) = \sum_{n=-\infty}^{\infty} h[Ln+\ell]\, e^{-j\omega n}$$

$$n' = Ln+\ell \Rightarrow n = \frac{n'-\ell}{L}$$

$$H_\ell(\omega) = \sum_{n'=Ln+\ell} h[n']\, e^{-j\omega\left(\frac{n'-\ell}{L}\right)}$$

$$= \left\{ \sum_{n'=-\infty}^{\infty} \frac{1}{L} \sum_{k=0}^{L-1} e^{j2\pi\frac{k}{L}(n'-\ell)}\, h[n']\, e^{-j\frac{\omega}{L}n'} \right\} e^{j\frac{\omega\ell}{L}}$$

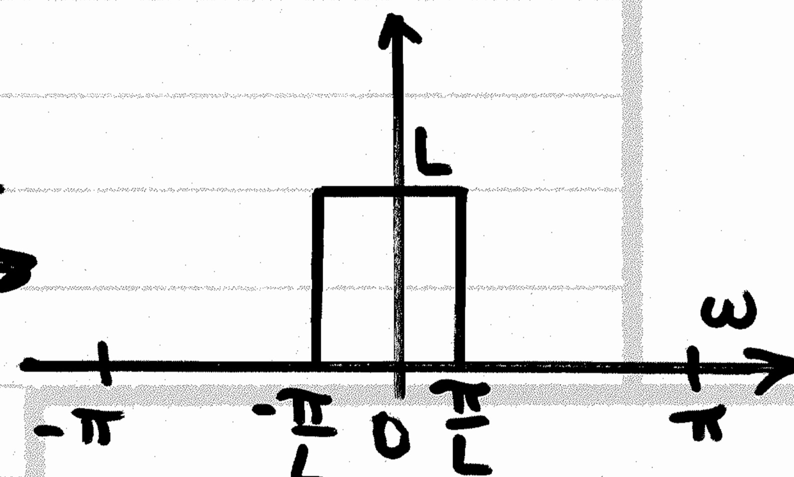
$$= \left\{ \sum_{k=0}^{L-1} \frac{1}{L} e^{-j2\pi\frac{\ell}{L}k} \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\frac{n}{L}(\omega-2\pi k)} \right\} e^{j\frac{\omega\ell}{L}}$$

$$H_\ell(\omega) = \left\{ \sum_{k=0}^{L-1} \frac{1}{L} e^{-j\frac{2\pi \ell k}{L}} H\left(\frac{\omega - k2\pi}{L}\right) \right\} e^{j\omega\frac{\ell}{L}}$$

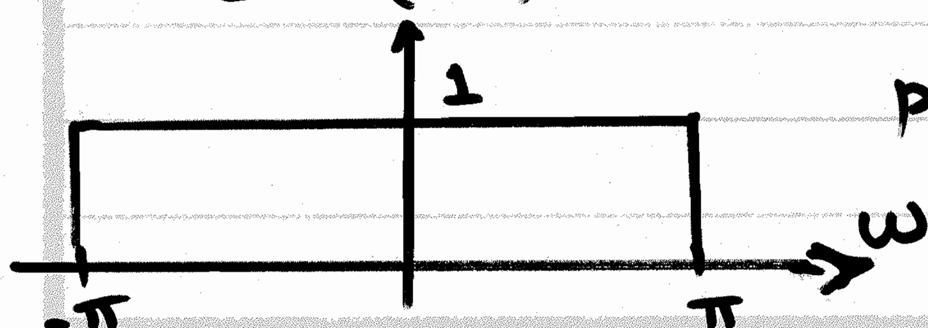
is the DTFT of $h[Ln+\ell]$

- Consider special case:

$$h[n] = L\,\frac{\sin\left(\frac{\pi}{L}n\right)}{\pi n} \xleftrightarrow{\text{DTFT}}$$



Then: $\frac{1}{L} H\left(\frac{\omega}{L}\right)$



period $= L\,2\pi$

The other values of k don't contribute in $-\pi<\omega<\pi$ $\Rightarrow$ just serve to make $H_\ell(\omega)$ be periodic with period 2π

Thus, for this special case, for $|\omega|<\pi$

$$H_\ell(\omega) = e^{j\omega \frac{\ell}{L}} \quad \text{for } |\omega|<\pi$$

recall: $x[n-n_0] \overset{DTFT}{\longleftrightarrow} X(\omega)e^{-j\omega n_0}$

$= x_a(nT_s - n_0T_s)$

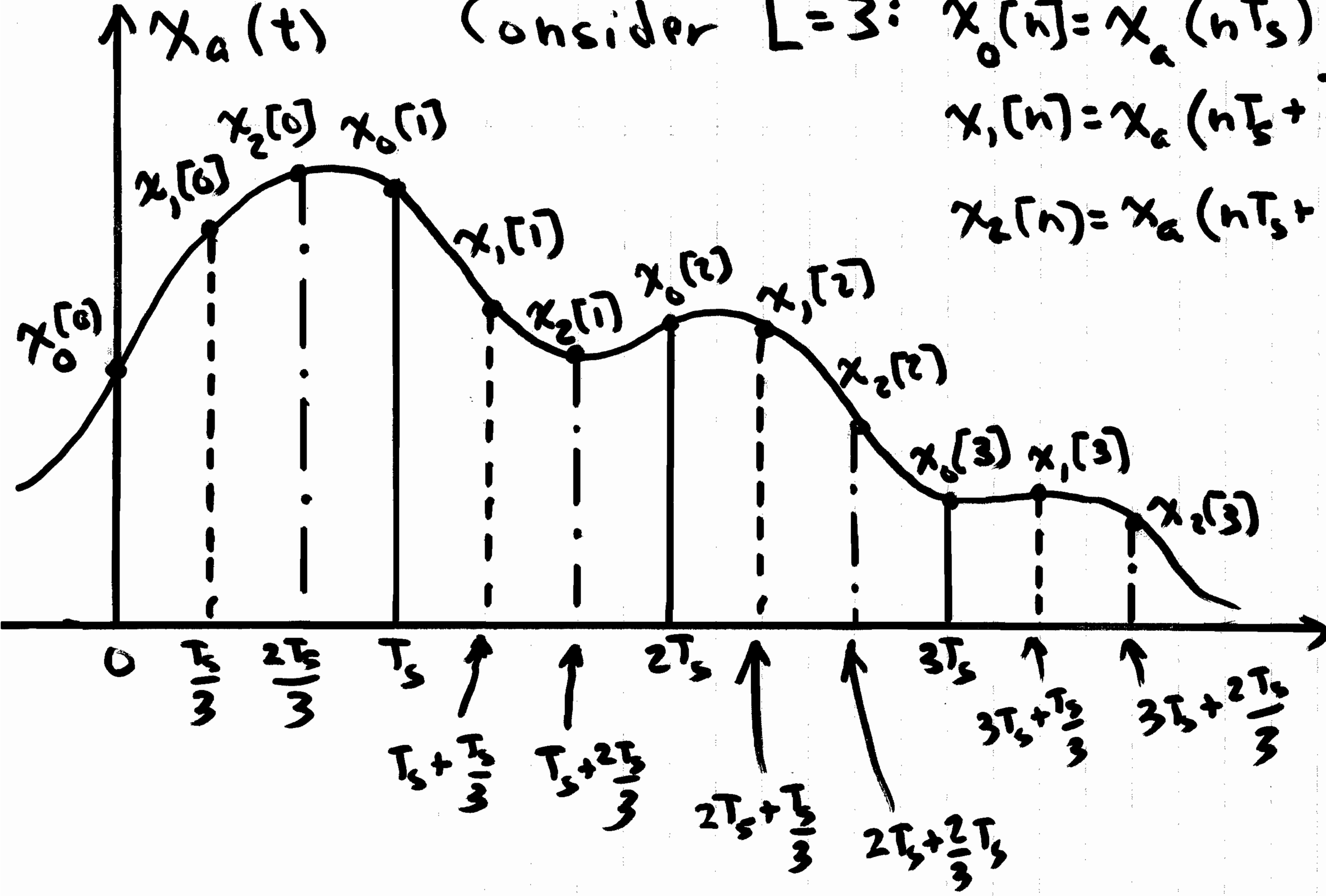
THUS: $e^{j\omega\frac{\ell}{L}}$ translates into a time-shift to the left of $\frac{\ell}{L}T_s$ in the analog domain

$\Rightarrow$ a fraction of a sample time-shift

Consider $L=3$: $x_0[n] = x_a(nT_s)$

$x_1[n] = x_a\left(nT_s + \frac{T_s}{3}\right)$

$x_2[n] = x_a\left(nT_s + 2\frac{T_s}{3}\right)$

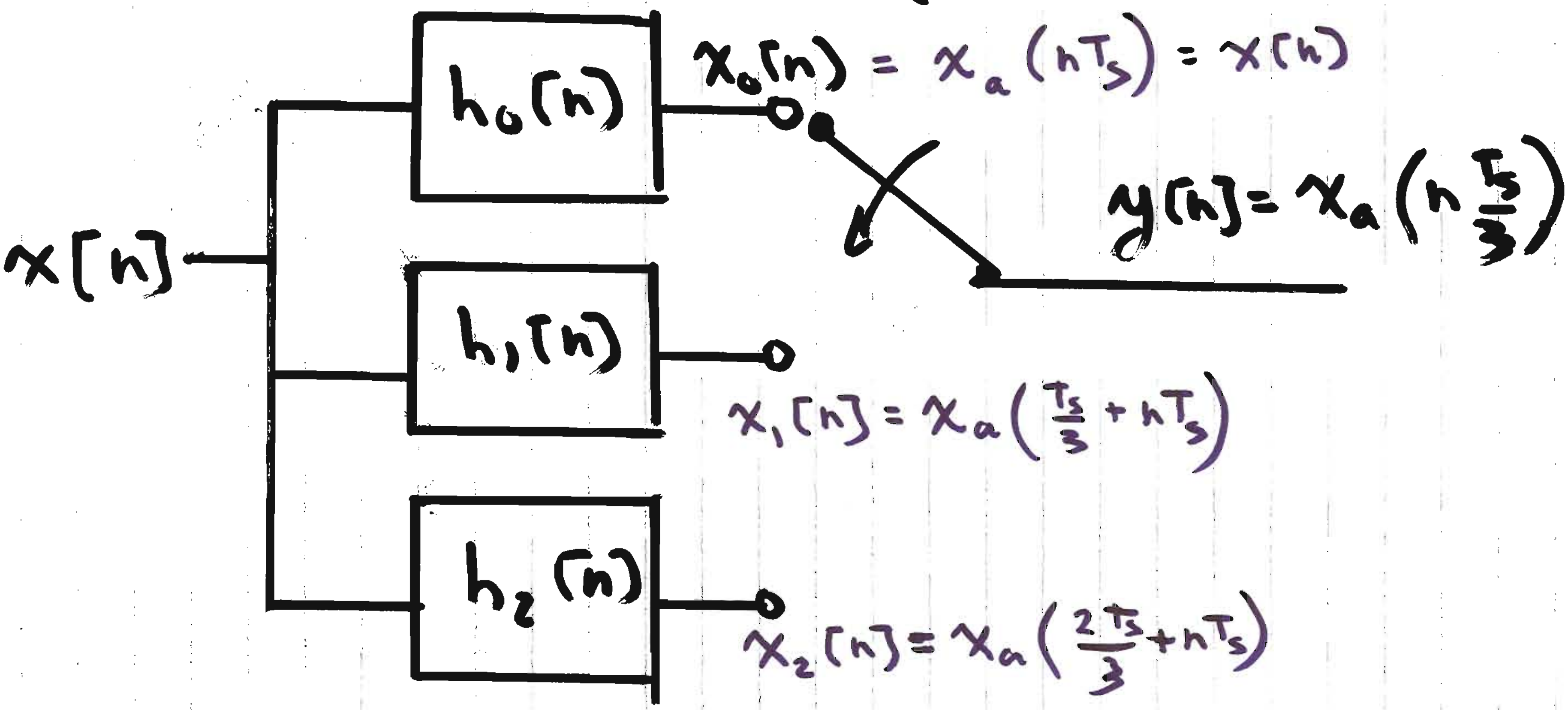


$$h[n] = 3\,\frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}$$

$$h_0[n] = h[Ln]$$

$$h_1[n] = h[Ln+1]$$

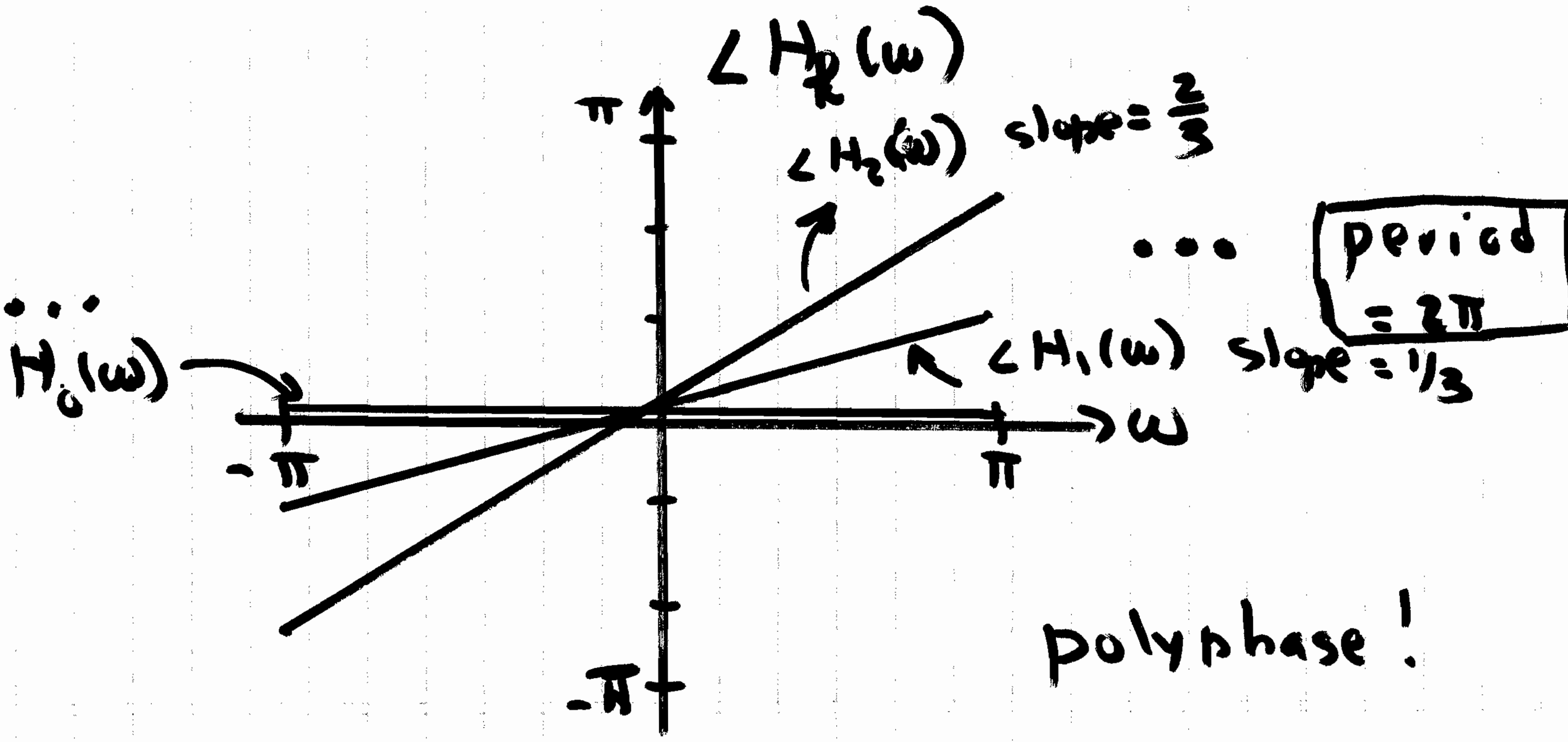
$$h_2[n] = h[Ln+2]$$



for $|\omega| < \pi$: $H_0(\omega) = 1$

$H_1(\omega) = e^{j\frac{1}{3}\omega} \Rightarrow \angle H_k(\omega) = \frac{\omega}{3}$

$H_2(\omega) = e^{j\frac{2}{3}\omega} \Rightarrow \angle H_2(\omega) = \frac{2}{3}\omega$



polyphase!

Note on Fractional Time-Shift

- Recall ideal Reconstruction Formula

$$x_a(t) = \sum_{k=-\infty}^{\infty} \underbrace{x_a(kT_s)}_{x[k]} \frac{\sin\left(\frac{\pi}{T_s}(t-kT_s)\right)}{\frac{\pi}{T_s}(t-kT_s)}$$

- Evaluate at: $t = \frac{l}{L}T_s + nT_s$ $\quad l \in \{0,1,\ldots,L-1\}$, $L = \text{integer}$

$$y[n] = x_a\left(\frac{l}{L}T_s + nT_s\right) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\frac{\pi}{T_s}\left(\frac{l}{L}T_s + nT_s - kT_s\right)\right)}{\frac{\pi}{T_s}\left(\frac{l}{L}T_s + nT_s - kT_s\right)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left(\pi\left(\frac{l}{L} + n - k\right)\right)}{\pi\left(\frac{l}{L} + n - k\right)}$$

$$= x[n] * h[n] \qquad h[n] = \frac{\sin\left(\pi\left(n + \frac{l}{L}\right)\right)}{\pi\left(n + \frac{l}{L}\right)}$$

$$= x_a\left(\frac{l}{L}T_s + nT_s\right)$$