EE538 Solution to Exam 3 Fall '99

Sol'n. to Prob. 1

$$x[n] = 31\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{DTFT} X(\omega) = \frac{31}{1-0.5e^{-j\omega}}$$

$$\sum_{\ell=-\infty}^{\infty} x[n+\ell N] = x_N[n] \underset{N}{\xleftrightarrow{DFT}} X_N(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}$$

$n=0,1,\ldots,N-1$

- for $N=5$:

$$x_N[n] = \sum_{\ell=-\infty}^{\infty} 31\left(\frac{1}{2}\right)^{n+\ell N} u[n+\ell N]$$

$n=0,1,\ldots,N-1$

$$= 31\sum_{\ell=0}^{\infty}\left(\frac{1}{2}\right)^{n+\ell N}$$

$$= 31\left(\frac{1}{2}\right)^n \sum_{\ell=0}^{\infty}\left(\left(\frac{1}{2}\right)^5\right)^{\ell}$$

$$= 31\left(\frac{1}{2}\right)^n \frac{1}{1-\frac{1}{32}}$$

$$= \left(\frac{1}{2}\right)^n 31 \frac{1}{31/32}$$

$$= 32\left(\frac{1}{2}\right)^n, \quad n=0,1,2,3,4$$

$$x_N[n] = \{\underset{\uparrow}{32}, 16, 8, 4, 2\}$$

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Sol'n. to Prob. (2)

(a) $w[n] = e^{j\frac{2\pi}{M}n}\left\{u[n] - u[n-\frac{M}{2}]\right\}$

$$* \, e^{-j\frac{2\pi}{M}n}\left\{u[n] - u[n-\frac{M}{2}]\right\}$$

• for $n = 0, 1, \ldots, \frac{M}{2} - 1$:

$$w[n] = \sum_{k=0}^{n} e^{j\frac{2\pi}{M}k} e^{-j\frac{2\pi}{M}(n-k)}$$

$$= e^{-j\frac{2\pi}{M}n} \sum_{k=0}^{n} e^{j\frac{4\pi}{M}k}$$

$$= e^{-j\frac{2\pi}{M}n} \frac{1 - e^{j\frac{4\pi}{M}(n+1)}}{1 - e^{j\frac{4\pi}{M}}}$$

$$= \frac{e^{-j\frac{2\pi}{M}n} e^{j\frac{2\pi}{M}(n+1)}}{e^{j\frac{2\pi}{M}}} = \frac{\sin\left(\frac{2\pi}{M}(n+1)\right)}{\sin\left(\frac{2\pi}{M}\right)}$$

$$w[n] = \frac{\sin\left[\frac{2\pi}{M}(n+1)\right]}{\sin\left[\frac{2\pi}{M}\right]} \quad \text{for } n = 0, 1, \ldots, \frac{M}{2} - 1$$

• for $n = \frac{M}{2}, \ldots, M-2$:

$$w[n] = \sum_{k = n - \frac{M}{2} + 1}^{M/2 - 1} e^{j\frac{2\pi}{M}k} e^{-j\frac{2\pi}{M}(n-k)}$$

Sol'n. to Prob. 2 (cont.)

- change of variables: $k' = k - \left(n - \frac{M}{2} + 1\right) = k - n + \frac{M}{2} - 1$ $\Big\}\ k = k' + n - \frac{M}{2} + 1$
  - limits: $k' \Big|_{0}^{\frac{M}{2} - 1 - n + \frac{M}{2} - 1 = M - n - 2}$

$$w[n] = \sum_{k'=0}^{M-2-n} e^{j\frac{4\pi}{M}\left(k' + n - \frac{M}{2} + 1\right)} \cdot e^{-j\frac{2\pi}{M}n}$$

$$= e^{-j\frac{2\pi}{M}n} e^{j\frac{4\pi}{M}n} e^{-j2\pi} e^{j\frac{4\pi}{M}} \sum_{k=0}^{M-2-n} e^{j\frac{4\pi}{M}k}$$

$$= e^{j\frac{2\pi}{M}n} e^{j\frac{4\pi}{M}} \frac{1 - e^{j\frac{4\pi}{M}(M-1-n)}}{1 - e^{j\frac{4\pi}{M}}}$$

$$= e^{j\frac{2\pi}{M}n} e^{j\frac{4\pi}{M}} \frac{e^{-j\frac{2\pi}{M}(n+1)}}{e^{j\frac{2\pi}{M}}} \frac{(-1)\sin\left(\frac{2\pi}{M}(n+1)\right)}{\sin\left(\frac{2\pi}{M}\right)}$$

* ANSWER 2(a) * *

$$= -\frac{\sin\left(\frac{2\pi}{M}(n+1)\right)}{\sin\left(\frac{2\pi}{M}\right)} \quad \text{for} \quad n = \frac{M}{2}, \frac{M}{2}+1, \ldots, M-2$$

- from before, for $n = 0, 1, \ldots, \frac{M}{2} - 1$:

$$w[n] = \frac{\sin\left(\frac{2\pi}{M}(n+1)\right)}{\sin\left(\frac{2\pi}{M}\right)}$$

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Sol'n. to Prob. 2

(b) $w[n] = w[M-2-n]$ $\Big\}$ Symmetric!

$n = 0, 1, \ldots, M-2$

Proof:

$$\frac{\sin\left[\frac{2\pi}{M}(M-2-n+1)\right]}{\sin\left(\frac{2\pi}{M}\right)} = \frac{\sin\left[2\pi - \frac{2\pi}{M}(n+1)\right]}{\sin\left(\frac{2\pi}{M}\right)}$$

$$= -\frac{\sin\left[\frac{2\pi}{M}(n+1)\right]}{\sin\left(\frac{2\pi}{M}\right)}$$

Hence, the negative sign for the expression for $w[n]$ for $n = \frac{M}{2}, \frac{M}{2}+1, \ldots, M-2$

(c) See plot attached.

$$W(\omega) = \frac{\sin\left[\frac{M}{4}\left(\omega - \frac{2\pi}{M}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega - \frac{2\pi}{M}\right)\right]} \cdot \frac{\sin\left[\frac{M}{4}\left(\omega + \frac{2\pi}{M}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega + \frac{2\pi}{M}\right)\right]}$$

nulls at $\omega = \frac{2\pi}{M} + \ell\frac{4\pi}{M}, \quad \ell = 0, 1, \ldots, \frac{M}{4} - 1$

and at $\quad = -\frac{2\pi}{M} - \ell\frac{4\pi}{M}, \quad$ " "

(d) mainlobe width: $\frac{4\pi}{M}$ (null-to-null)

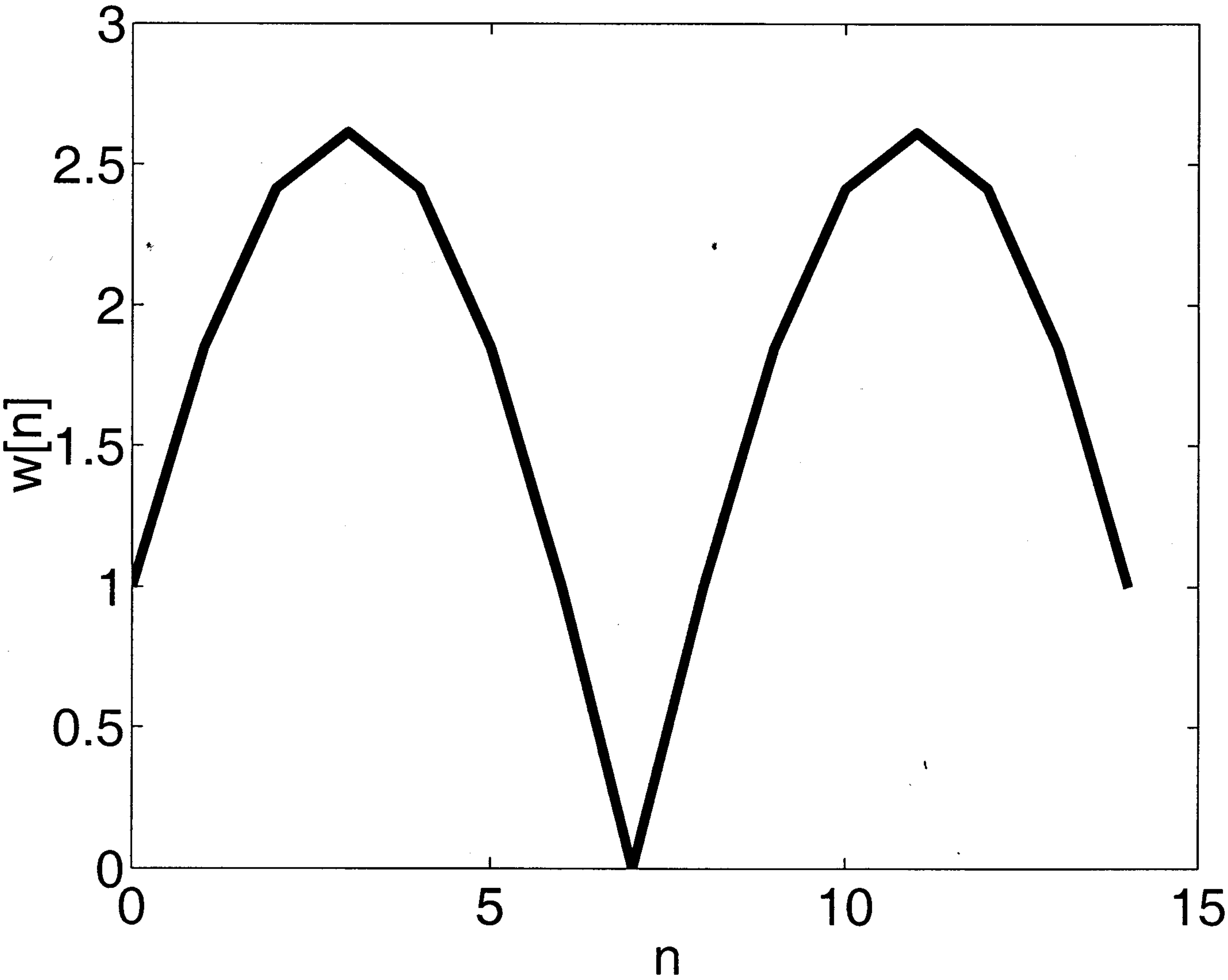
This is exactly the same as the mainlobe width for a rectangular window of length M-1

Sol'n. to Prob. 2 (cont.)

Part (e). The peak sidelobe of the "new" window is larger than the peak sidelobe for a rectangular window of the same length $M-1$. This is due to the sharp discontinuity in the derivative of the window at the middle point $n = \frac{M}{2} - 1$. See the plot attached.

Part (f). Because the mainlobe width is the same and since there is more energy in the two peak sidelobes, there is less energy in the other sidelobes. Thus, the "other" sidelobes are lower than those for a rectangular window of the same length. See the plot attached.

w[n] for M = 16



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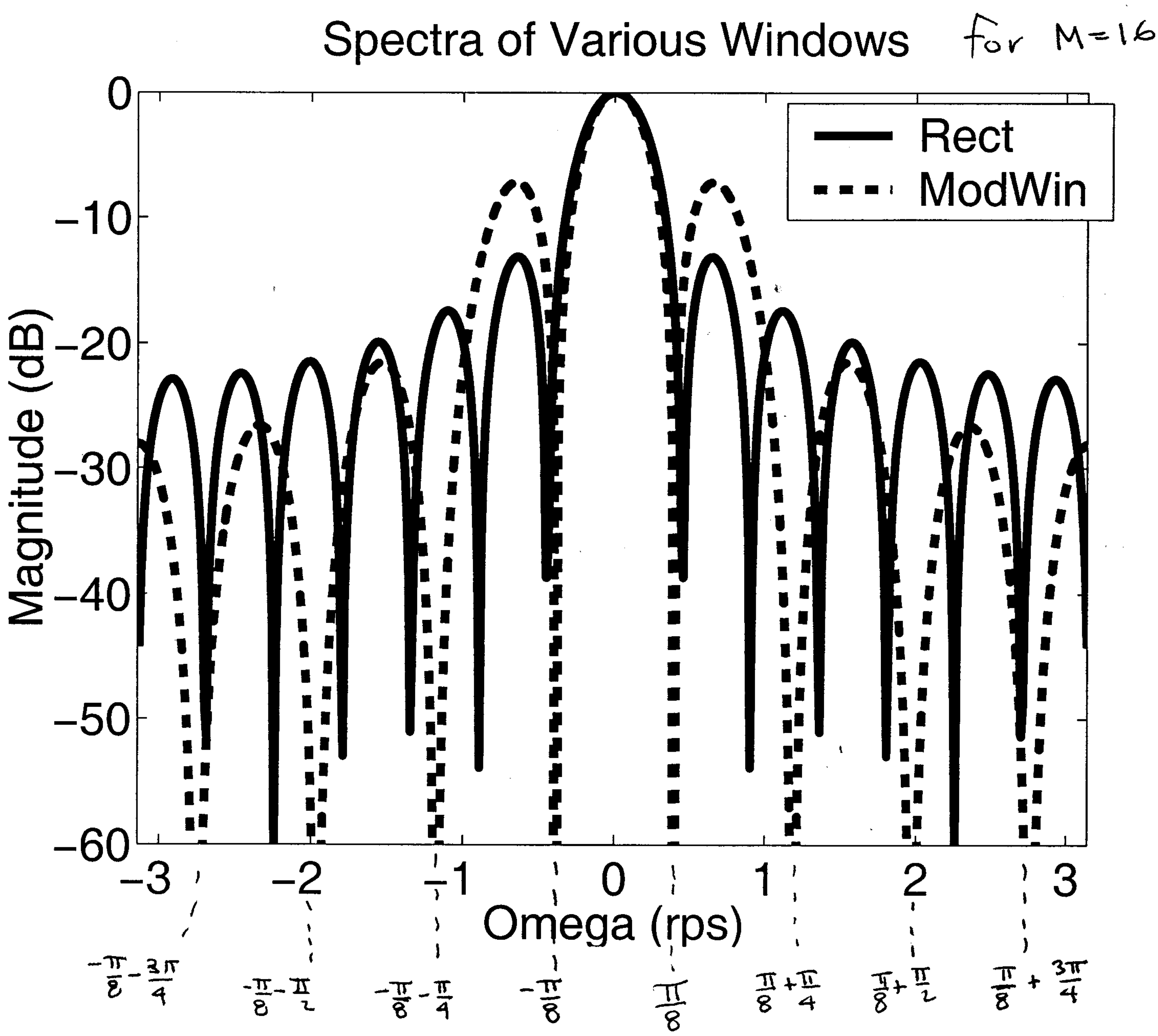
Solution to Prob. 2 (cont.)

Answer to Part (c):



nulls at $\omega = \frac{2\pi}{16} + \ell\frac{4\pi}{16}$, $\ell = 0,1,2,3$

and $\omega = -\frac{2\pi}{16} - \ell\frac{4\pi}{16}$, $\ell = 0,1,2,3$

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Sol'n. to Prob. 3 : (Sol'n. to Prob. 2 follows)

Given: $r_{xx}[0]=2$; $r_{xx}[1]=1$; $r_{xx}[2]=-1$

(a)

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

A $\quad 2a_1 + a_2 = -1$

B $\quad a_1 + 2a_2 = 1$

-2 A $+$ B $\Rightarrow (-4+1)a_1 = 2+1 = 3$

$$a_1 = -1$$

$$a_2 = -1 - 2a_1 = -1 - 2(-1) = 1 \quad \left.\right\} \; a_1 = -1,\; a_2 = 1$$

(b) $r_{xx}[3] = -a_1 r_{xx}[2] - a_2 r_{xx}[1]$

$$r_{xx}[3] = r_{xx}[2] - r_{xx}[1]$$

$$= -1 - 1 = -2$$

(c) $S_{xx}(\omega) = ?$ $\qquad r_{xx}[m] = \left(\frac{A}{\sqrt{2}}\right)^2 \cos(\omega_0 m)$

$$S_{xx}(\omega) = \frac{A^2}{2}\left\{\pi\delta(\omega-\omega_0) + \pi\delta(\omega-\omega_0)\right\}$$

$\omega_0 = ?$ $\quad\Rightarrow\quad z^2 + a_1 z + a_2 = z^2 - z + 1$

Answer: $\omega_0 = \frac{\pi}{3}$

$$= (z - e^{j\omega_0})(z - e^{-j\omega_0})$$

$$= (z - e^{j\frac{\pi}{3}})(z - e^{-j\frac{\pi}{3}})$$