Derivation of Levinson-Durbin Algorithm:

- m-th order predictor:

$$\underline{R}_m \underline{a}_m = -\underline{r}_m = -\begin{bmatrix} r[1] \\ \vdots \\ r[m] \end{bmatrix} = -\begin{bmatrix} \underline{r}_{m-1} \\ r[m] \end{bmatrix}$$

- for order m-1:

$$\underline{R}_{m-1} \underline{a}_{m-1} = -\underline{r}_{m-1}$$

$$\underline{R}_m = \begin{bmatrix} r[0] & \underline{r}_{m-1}^T \\ \underline{r}_{m-1} & \underline{R}_{m-1} \end{bmatrix}$$

$$\underline{a}_m = -\underline{R}_m^{-1} \underline{r}_m$$

$$\underline{a}_{m-1} = -\underline{R}_{m-1}^{-1} \underline{r}_{m-1}$$

Partitioned matrix inversion lemma:

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, then

$$A^{-1} = \left[\begin{array}{c|c} E^{-1} & -E^{-1}A_{12}A_{22}^{-1} \\ \hline -A_{22}^{-1}A_{21}E^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}E^{-1}A_{12}A_{22}^{-1} \end{array}\right]$$

- where: $E = A_{11} - A_{12}A_{22}^{-1}A_{21}$
- Here: partition $\underline{R}_m$ such that $A_{11} = r[0]$ so that E is 1×1 scalar $E^{-1} = 1/E$ and $A_{22} = \underline{R}_{m-1}$

- Examine E:

$$E = r[0] - \underline{r}_{m-1}^{T} \underline{R}_{m-1}^{-1} \underline{r}_{m-1}$$

$$= r[0] - \underline{r}_{m-1}^{T} (-\underline{a}_{m-1})$$

$$= r[0] + \underline{r}_{m-1}^{T} \underline{a}_{m-1}$$

$$= r[0] + \sum_{k=1}^{m-1} a_{m-1}(k)\, r_{xx}[k]$$

$\left(= r_{xx}^{*}[k] \text{ for real-valued } x[n] \right)$

$$= \mathcal{E}_{m-1}^{min}$$

$$E^{-1} = 1/\mathcal{E}_{m-1}^{min}$$

$$\underline{R}_m^{-1} = \frac{1}{\varepsilon_{m-1}^{\min}} \left[\begin{array}{c|c} 1 & -\underline{r}_{m-1}^T \underline{R}_{m-1}^{-1} \\ \hline -\underline{R}_{m-1}^{-1} \underline{r}_{m-1} & \varepsilon_{m-1}^{\min} \underline{R}_{m-1}^{-1} + \underline{R}_{m-1}^{-1} \underline{r}_{m-1} \underline{r}_{m-1}^T \underline{R}_{m-1}^{-1} \end{array} \right]$$

$$= \frac{1}{\varepsilon_{m-1}^{\min}} \left[\begin{array}{c|c} 1 & \underline{a}_{m-1}^T \\ \hline +\underline{a}_{m-1} & \varepsilon_{m-1}^{\min} \underline{R}_{m-1}^{-1} + \underline{a}_{m-1} \underline{a}_{m-1}^T \end{array} \right]$$

Recall reverse permutation matrix $\tilde{I}$ satisfies $\tilde{I}\,\tilde{I} = I$

Also: $\tilde{I}\,\underline{R}_m\,\tilde{I} = \underline{R}_m$ (for real-valued $x[n]$)

Trick: pre-multiply both sides of $\underline{a}_m = -\underline{R}_m^{-1}\underline{r}_m$ by $\tilde{I}$:

$$\tilde{I}\underline{a}_m = -\tilde{I}\underline{R}_m^{-1}\tilde{I}\tilde{I}\underline{r}_m$$

Since $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ and $\tilde{I}^{-1} = \tilde{I}$

$$\underline{R}_m^{-1} = (\tilde{I}\underline{R}_m\tilde{I})^{-1} = \tilde{I}\underline{R}_m^{-1}\tilde{I} = \underline{R}_m^{-1}$$

$$\tilde{I}\underline{a}_m = \begin{bmatrix} a_m(m) \\ \tilde{I}\underline{a}_m(1:m-1) \end{bmatrix} = -\underline{R}_m^{-1}\begin{bmatrix} r[m] \\ \tilde{I}\underline{r}_{m-1} \end{bmatrix}$$

Substituting expression for $\underline{R}_m^{-1}$:

$$\begin{bmatrix} a_m(m) \\ \hline \tilde{I}\,\underline{a}_m(1:m-1) \end{bmatrix} = \frac{-1}{\mathcal{E}_{m-1}^{min}} \begin{bmatrix} r[m] + \underline{a}_{m-1}^T \tilde{I}\,\underline{r}_{m-1} \\ \hline r[m]\,\underline{a}_{m-1} + (\underline{a}_{m-1}^T \tilde{I}\,\underline{r}_{m-1})\,\underline{a}_{m-1} \\ + \mathcal{E}_{m-1}^{min}\,\underline{R}_{m-1}^{-1}\,\tilde{I}\,\underline{r}_{m-1} \end{bmatrix}$$

Thus: $$a_m(m) = -\frac{\left\{ r[m] + \underline{a}_{m-1}^T \tilde{I}\,\underline{r}_{m-1} \right\}}{\mathcal{E}_{m-1}^{min}}$$

and:

$$\underline{a}_m(1:m-1) = +\,a_m(m)\,\tilde{I}\,\underline{a}_{m-1} - \tilde{I}\,\underline{R}_{m-1}^{-1}\,\tilde{I}\,\underline{r}_{m-1}$$

$$= a_m(m)\,\tilde{I}\,\underline{a}_{m-1} - \underline{a}_{m-1}$$

- End of Derivation